INTEGRABLE SYSTEMS AND MODULI SPACES OF CURVES

PAOLO ROSSI

Mémoire d'Habilitation à Diriger des Recherches
soutenue le 28/11/2016 à l'Université de Bourgogne
Ecole doctorale Carnot-Pasteur

Composition du Jury:
Vladimir Roubtsov, président du Jury, Professeur,
Ugo Bruzzo, Professeur,
Alessandro Chiodo, Professeur,
Boris Dubrovin, Professeur,
Daniele Faenzi, Professeur,
Nikolai Kitanine, Professeur,
Dimitri Zvonkine, Chargé de Recherches

Rapporteurs:
Alessandro Chiodo, Professeur,
Yakov Eliashberg, Professeur,
Rahul Pandharipande, Professeur,
Dimitri Zvonkine, Chargé de Recherches

Abstract

This document has the purpose of presenting in an organic way my research on integrable systems originating from the geometry of moduli spaces of curves, with applications to Gromov-Witten theory and mirror symmetry. The text contains a short introduction to the main ideas and prerequisites of the subject from geometry and mathematical physics, followed by a synthetic review of some of my papers (listed below) starting from my PhD thesis (October 2008), and with some open questions and future developements.

My results include:

- the triple mirror symmetry among $\mathbb{P}^1$-orbifolds with positive Euler characteristic, the Landau-Ginzburg model with superpotential $-xyz + x^p + y^q + z^r$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ and the orbit spaces of extended affine Weyl groups of type ADE,
- the mirror symmetry between local footballs (local toric $\mathbb{P}^1$-orbifolds) and certain double Hurwitz spaces together with the identification of the corresponding integrable hierarchy as a rational reduction of the 2DToda hierarchy (with A. Brini, G. Carlet and S. Romano).
- a series of papers on various aspects of the double ramification hierarchy (after A. Buryak), forming a large program investigating integrable systems arising from cohomological field theories and the geometry of the double ramification cycle, their quantization, their relation with the Dubrovin-Zhang hierarchy, the generalizations of Witten's conjecture and relations in the cohomology of the moduli space of stable curves (with A. Buryak, B. Dubrovin and J. Guéré).

In section 1 we discuss the notions of Hamiltonian systems of evolutionary PDEs, integrable systems and tau-structure, with the aim of giving a quick yet somewhat precise introduction to the mathematical physics needed for understanding Witten's conjecture and its generalizations.

In section 2, after a short recall of the basic objects from the geometry of the moduli space of stable curves, we review the definition of cohomological field theories and the double ramification cycle.

In section 3 we recall the construction of the Dubrovin-Zhang hierarchy, an integrable system associated to a semisimple cohomological field theory. Using this language we formulate Witten's conjecture and its generalizations.

In section 4 we quickly review the Frobenius manifold approach to mirror symmetry.

Sections 5 and 6 are devoted to describing my two main research lines and the content of a selection of my papers. Section 5 deals with quantum cohomology, mirror symmetry and integrable systems. Section 6 deals with the more recent project of constructing, understanding and computing the double ramification hierarchy, a novel way of associating an integrable system to a given cohomological field theory.

List of papers concerned by this exposition

P. ROSSI, *Symplectic Topology, Mirror Symmetry and Integrable Systems*, PhD Thesis, defended at SISSA (Trieste) on 21/10/2008 .

P. ROSSI, *Gromov-Witten invariants of target curves via Symplectic Field Theory*, Journal of Geometry and Physics, Volume 58, Issue 8, August 2008, Pages 931-941.

P. ROSSI, *Gromov-Witten theory of orbicurves, the space of tri-polynomials and Symplectic Field Theory of Seifert fibrations*, Mathematische Annalen (2010) 348:265-287.

P. ROSSI, *Integrable systems and holomorphic curves*, Gokova Geometry and Topology 2009, International Press, April 2010.

O. FABERT, P. ROSSI, *String, dilaton and divisor equation in Symplectic Field Theory*, International Mathematics Research Notices, Volume 2011, Issue 19, pp. 4384-4404.

A. BRINI, G. CARLET, P. ROSSI, *Integrable hierarchies and the mirror model of local* $\mathbb{CP}^1$, Physica D 241 (2012) 2156-2167.

A. BRINI, G. CARLET, S. ROMANO, P. ROSSI, *Rational reductions of the 2D-Toda hierarchy and mirror symmetry*, accepted in the Journal of the European Mathematical Society (JEMS) - preprint arXiv:1401.5725.

A. BURYAK, P. ROSSI, *Recursion relations for Double Ramification Hierarchies*, Communications in Mathematical Physics, March 2016, Volume 342, Issue 2, pp 533-568.

A. BURYAK, P. ROSSI, *Double Ramification Cycles and Quantum Integrable Systems*, Letters in Mathematical Physics, March 2016, Volume 106, Issue 3, pp 289-317.

A. BURYAK, B. DUBROVIN, J. GUÉRÉ, P. ROSSI, *Tau-structure for the Double Ramification Hierarchies*, submitted, preprint arXiv:1602.05423.

A. BURYAK, B. DUBROVIN, J. GUÉRÉ, P. ROSSI, *Integrable systems of double ramification type*, preprint arXiv:1609.04059.

Contents

1. Integrable systems

In this section I will try to give, in a few pages, a precise idea of what an integrable system is, in the context of evolutionary Hamiltonian PDEs. We will introduce the minimal notions that will be used in what follows and assume a certain familiarity with the finite dimensional theory of Poisson manifolds, to guide the reader in extending such notions to an infinite-dimensional context.

1.1. **Formal loop space.** An evolutionary PDE is a system of differential equations of the form

$$\partial_t u^\alpha = F^\alpha(u^*, u_1^*, u_2^*, \ldots), \quad \alpha = 1, \ldots, N$$

where $u_k^\alpha = \partial_x^k u^\alpha$ and we use the symbol $*$ to indicate any value for the sub or superscripts.

Such a system can be heuristically interpreted as a vector field on the infinite-dimensional space of all loops $u\colon S^1 \to V$, where V is a N-dimensional vector space with a basis $e_1, \ldots, e_N$ and x is the coordinate on S^1, so that $u^\alpha = u^\alpha(x)$ is the component along e_α of such loop. This is just a heuristic interpretation as we choose to work in a more formal algebraic setting by describing an appropriate ring of functions for the loop space of V as follows.

Consider the ring of differential polynomials $\widehat{\mathcal{A}} = \mathbb{C}[[u^*]][u^*_{>0}][[\varepsilon]]$ and endow it with the grading $\deg(u_k^\alpha) = k$, $\deg(\varepsilon) = -1$. The role of the parameter ε and grading will become clear shortly. The operator ∂_x acts on $\widehat{\mathcal{A}}$ in the obvious way, i.e. $\partial_x = \sum_{k\geq 0} u^\alpha_{k+1} \frac{\partial}{\partial u^\alpha_k}$ (we use the convention of sum over repeated greek indices, but not roman indices).

We define the space of local functionals as the quotient $\widehat{\Lambda} = \widehat{\mathcal{A}}/(\mathrm{Im}\partial_\mathrm{x} \oplus \mathbb{C})$. The equivalence class of $f(u_*^*;\varepsilon) \in \widehat{\mathcal{A}}$ in this quotient will be denoted suggestively as $\overline{f} = \int f(u^*_*;\varepsilon)dx$ (hinting at the quotient with respect to $\mathrm{Im}\partial_\mathrm{x}$ as the possibility of integrating by parts on the circle S^1).

Local functionals in $\widehat{\Lambda}$ can hence be interpreted as functions on our formal loop space of V whose value on a given loop $u : S^1 \to V$ is the integral over S^1 of some differential polynomial in its components $u^\alpha(x)$.

Changes of coordinates on the formal loop space will be described accordingly as

$$\widetilde{u}^\alpha = \widetilde{u}^\alpha(u^*_*, \varepsilon) \in \widehat{\mathcal{A}}^{[0]}, \qquad \det\left(\frac{\partial \widetilde{u}^*|_{\varepsilon=0}}{\partial u^*}\right) \neq 0.$$

Notice here the importance of the parameter ε, whose exponent counts the number of x-derivatives appearing in $\widetilde{u}^\alpha$. Its importance lies in the fact that we can use the parameter ε to invert such change of coordinates: for fixed $\widetilde{u}^\alpha(x)$, we just need to solve the ODE $\widetilde{u}^\alpha = \widetilde{u}^\alpha(u^*_*, \varepsilon)$ for the functions $u^\alpha(x)$ order by order in ε and we will obtain a differential polynomial $u^\alpha = u^\alpha(\widetilde{u}^*_*;\varepsilon)$. The resulting group is called the Miura group.

Differential polynomials and local functionals can also be described using another set of formal variables, corresponding heuristically to the Fourier components p_k^α,

$k \in \mathbb{Z}$, of the functions $u^\alpha = u^\alpha(x)$. Let us, hence, define a change of variables

$$u^\alpha_j = \sum_{k\in\mathbb{Z}} (ik)^j p^\alpha_k e^{ikx}, \tag{1.1}$$

which is nothing but the j-th derivative of $u^\alpha = \sum_{k\in\mathbb{Z}} p^\alpha_k e^{ikx}$.

This allows us to express a differential polynomial $f(u; u_x, u_{xx}, \dots; \varepsilon) \in \widehat{\mathcal{A}}^{[d]}$ as a formal Fourier series $f = \sum f^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n;s} \varepsilon^s p^{\alpha_1}_{k_1} \dots p^{\alpha_n}_{k_n} e^{i(\sum_{j=1}^n k_j)x}$ where the coefficient $f^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n;s}$ is a polynomial in the indices $k_1, \dots, k_n$ of degree $s + d$. Moreover, the local functional $\overline{f}$ corresponds to the constant term of the Fourier series of f.

1.2. **Poisson structures.** In what follows we will be interested in Hamiltonian systems of evolutionary PDEs. To this end we endow the space of local functionals with a Poisson structure of the form

$$\{\overline{f}, \overline{g}\}_K := \int \frac{\delta \overline{f}}{\delta u^\mu} K^{\mu\nu} \frac{\delta \overline{g}}{\delta u^\nu} dx,$$

where $K^{\mu\nu} = \sum_{j\geq 0} K^{\mu\nu}_j \partial^j_x$, $K^{\mu\nu}_j \in \widehat{\mathcal{A}}^{[-j+1]}$. Given that the variational derivative $\frac{\delta}{\delta u^\alpha} = \sum_{k\geq 0} (-\partial_x)^k \frac{\partial}{\partial u^\alpha_k}$ is the natural extension to local functionals of the finite dimensional notion of partial derivative, the above formula seems quite natural. The differential operator K is called a Hamiltonian operator. Imposing antisymmetry and the Jacobi identity for the Poisson brackets obviously imposes conditions on the differential operator $K^{\mu\nu}$. For instance

$$K^{\alpha\beta}|_{\varepsilon=0} = g^{\alpha\beta}(u)\partial_x + b^{\alpha\beta}_\gamma(u) u^\gamma_x, \tag{1.2}$$

and the matrix $(g^{\alpha\beta})$ is symmetric (and, for simplicity, we will always assume it nondegenerate), the inverse matrix $(g_{\alpha\beta})$ defines a flat metric and the functions $\Gamma^\gamma_{\alpha\beta}(u) := -g_{\alpha\mu}(u) b^{\mu\gamma}_\beta(u)$ are the coefficients of the Levi-Civita connection corresponding to this metric (see [DN83]).

We also define the Poisson bracket between a differential polynomial $f \in \widehat{\mathcal{A}}$ and a local functional $\overline{g} \in \widehat{\Lambda}$ as follows

$$\{f, \overline{g}\}_K = \sum_{s\geq 0} \frac{\partial f}{\partial u^\mu_s} \partial^s_x \left(K^{\mu\nu} \frac{\delta \overline{g}}{\delta u^\nu} \right)$$

Such formula, is compatible with the previous one in the sense that $\int \{f, \overline{g}\}_K dx = \{\overline{f}, \overline{g}\}_K$.

The action of a Miura transformation on the Poisson structure is given in terms of Hamiltonian operators as follows

$$K^{\alpha\beta}_{\tilde{u}} = (L^*)^\alpha_\mu \circ K^{\mu\nu}_u \circ L^\beta_\nu$$

where $(L^*)^\alpha_\mu = \sum_{s\geq 0} \frac{\partial \tilde{u}^\alpha}{\partial u^\mu_s} \partial^s_x$, $L^\beta_\nu = \sum_{s\geq 0} (-\partial_x)^s \circ \frac{\partial \tilde{u}^\beta}{\partial u^\nu_s}$.

The following Darboux-type theorem states that, up to change of coordinates, there exists but one Poisson structure on the formal loop space.

Theorem 1.1 ([Get02]). *There exist a Miura transformation bringing any Poisson bracket to the standard form*

$$K^{\mu\nu} = \eta^{\mu\nu} \partial_x, \qquad \eta^{\mu\nu} \text{ constant, symmetric and nondegenerate}$$

The standard Poisson bracket also has a nice expression in terms of the variables p^α_k:

$$\{p^\alpha_k, p^\beta_j\}_{\eta\partial_x} = ik\eta^{\alpha\beta}\delta_{k+j,0}.$$

1.3. **Integrable hierarchies.** A Hamiltonian system is an evolutionary PDE of the form

$$\partial_t u^\alpha = \{u^\alpha, \overline{h}\}_K = K^{\alpha\nu}\frac{\delta\overline{h}}{\delta u^\nu}, \qquad \overline{h}\in\widehat{\Lambda}^{[0]},$$

where $\overline{h}$ is called the Hamiltonian of the system.

An integrable system, or an integrable hierarchy, is an infinite system of Hamiltonian evolutionary PDEs

$$\partial_{t^\beta_d} u^\alpha = \{u^\alpha, \overline{h}_{\beta,d}\}_K = K^{\alpha\mu}\frac{\delta\overline{h}_{\beta,d}}{\delta u^\mu}, \qquad \overline{h}_{\beta,d}\in\widehat{\Lambda}^{[0]}, \tag{1.3}$$

generated by Hamiltonians $\overline{h}_{\alpha,d}\in\widehat{\Lambda}^{[0]}$, $\alpha=1,\dots,N$, $d\geq 0$ such that

$$\{\overline{h}_{\alpha,i}, \overline{h}_{\beta,j}\}_K = 0$$

As in the finite dimensional situation, the above Poisson-commutativity condition for the Hamiltonians is equivalent to the compatibility of the infinite system of PDEs they generate. A formal solution to the above integrable hierarchy is given by a formal power series $u^\alpha(x,t^*_*;\varepsilon)\in\mathbb{C}[[x,t^*_*,\varepsilon]]$ satisfying all the equations of the hierarchy simultaneously.

1.4. **Tau-functions.** Consider the Hamiltonian system (1.3). Let us assume that the Hamiltonian $\overline{h}_{1,0}$ generates the spatial translations:

$$\partial_{t^1_0} u^\alpha = K^{\alpha\mu}\frac{\delta\overline{h}_{1,0}}{\delta u^\mu} = u^\alpha_x.$$

A *tau-structure* for the hierarchy (1.3) is a collection of differential polynomials $h_{\beta,q}\in\widehat{\mathcal{A}}^{[0]}_N$, $1\leq\beta\leq N$, $q\geq -1$, such that the following conditions hold:

(1) $K^{\alpha\mu}\frac{\delta\overline{h}_{\beta,-1}}{\delta u^\mu}=0, \qquad \beta=1,\dots,N.$
(2) The N functionals $\overline{h}_{\beta,-1}$ are linearly independent.
(3) $\overline{h}_{\beta,q}=\int h_{\beta,q}dx, \qquad q\geq 0.$
(4) Tau-symmetry: $\frac{\partial h_{\alpha,p-1}}{\partial t^\beta_q}=\frac{\partial h_{\beta,q-1}}{\partial t^\alpha_p}, \quad 1\leq\alpha,\beta\leq N, \quad p,q\geq 0.$

Existence of a tau-structure imposes non-trivial constraints on a Hamiltonian hierarchy. A Hamiltonian hierarchy with a fixed tau-structure will be called *tau-symmetric.*

The fact that $\{\overline{h}_{\alpha,p-1},\overline{h}_{\beta,q}\}=0$ implies $\int\frac{\partial h_{\alpha,p-1}}{\partial t^\beta_q}dx=0$. Thus, there exists a differential polynomial $\Omega_{\alpha,p;\beta,q}\in\widehat{\mathcal{A}}^{[0]}$ such that $\partial_x\Omega_{\alpha,p;\beta,q}=\frac{\partial h_{\alpha,p-1}}{\partial t^\beta_q}$ (and hence, in particular, $h_{\alpha,p-1}=\Omega_{\alpha,p;1,0}$).

Consider an arbitrary solution $u^\alpha=u^\alpha(x,t^*_*;\varepsilon)\in\mathbb{C}[[x,t^*_*,\varepsilon]]$ of our hierarchy (1.3). Tau-symmetry guarantees that there exists a function $F\in\mathbb{C}[[t^*_*,\varepsilon]]$ such that

$$(\Omega_{\alpha,p;\beta,q}(u(x,t;\varepsilon);u_x(x,t;\varepsilon),\dots))|_{x=0}=\frac{\partial^2F}{\partial t^\alpha_p\partial t^\beta_q}, \quad \text{for any } 1\leq\alpha,\beta\leq N \text{ and } p,q\geq 0.$$

The function $F(t^*_*;\varepsilon)$ is called the tau-function of the given solution (in fact, for historical reasons, the tau-function should correspond to the exponential of F, but we will ignore this distinction here, calling F tau-function indistinctly). Tau-symmetric hierarchies hence have the property that the evolution along a particular solution of any of their Hamiltonian densities is subsumed under one single function $F(t^*_*;\varepsilon)$.

Given a tau-structure, its system of normal coordinates is the system of coordinates $\widetilde{u}^\alpha = \eta^{\alpha\mu} h_{\mu,-1}(u^*_*;\varepsilon)$. The Hamiltonian operator takes the form $K^{\alpha\beta}_{\widetilde{u}} = \eta^{\alpha\beta}\partial_x + O(\varepsilon)$, η being a constant symmetric nondegenerate matrix.

A class of Miura transformations preserving the tau structure is given by *normal Miura transformations*. Let u^α already be normal coordinates and $\mathcal{F}(u^*_*;\varepsilon) \in \widehat{\mathcal{A}}^{[-2]}$. The normal Miura transformation generated by $\mathcal{F}$ is given by

$$\widetilde{u}^\alpha = u^\alpha + \eta^{\alpha\mu}\partial_x\{\mathcal{F}, \overline{h}_{\mu,0}\}_K.$$

Then the Hamiltonian densities $\widetilde{h}_{\beta,q} = h_{\beta,q} + \partial_x\{\mathcal{F}, \overline{h}_{\beta,q+1}\}_K$ form again a tau-structure and the coordinates $\widetilde{u}^\alpha$ are normal for it. Moreover, for any solution of the system, its tau-function changes in the following way under the normal Miura transformation:

$$\widetilde{F}(t^*_*;\varepsilon) = F(t^*_*;\varepsilon) + \mathcal{F}(u^*_*(x,t^*_*;\varepsilon);\varepsilon)|_{x=0}$$

1.5. **Example: the KdV hierarchy.** The Korteweg-de Vries equation is the most well known example of integrable Hamiltonian PDEs. It is defined on the formal loop space of a one-dimensional vector space $V = \mathbb{C}$, so we will suppress the greek indices in all the above notations. The metric on V is simply $\eta = 1$. The Poisson structure is given by the Hamiltonian operator $K = \partial_x$ (so it is in Getzler's standard form). The Hamiltonian is the following local functional in $\widehat{\Lambda}^{[0]}$:

$$\overline{h}_{\mathrm{KdV}} = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_2\right) dx.$$

We can hence compute the Hamiltonian flow, i.e. the KdV equation:

$$u_t = u u_1 + \frac{\varepsilon^2}{24} u_3.$$

The KdV equation is one of the flows of an integrable hierarchy. There are various ways to compute the other flows (or the other Hamiltonians) which compose such hierarchy (see for instance [Dic03]). Here I choose to construct them by a recursive procedure that we discovered with A. Buryak in [BR16a] and which was not known before.

Let $g_{-1} = u \in \widehat{\mathcal{A}}^{[0]}$ and construct $\overline{h}_i \in \widehat{\Lambda}^{[0]}$, $i \geq -1$ as $\overline{h}_i = \int g_i dx$, where the differential polynomials $g_i \in \widehat{\mathcal{A}}^{[0]}$ are produced by the recursive equation

$$g_{i+1} = (D-1)^{-1}\partial_x^{-1}\{g_i, \overline{h}_{\mathrm{KdV}}\}, \quad D := \sum_{k\geq 0}(k+1)u_k\frac{\partial}{\partial u_k}.$$

At each step, this equation produces a new Hamiltonian density whose Poisson bracket with $\overline{h}_{\mathrm{KdV}} = \overline{h}_1$ is ∂_x-exact so that it makes sense to take the inverse x-derivative. The operator $D-1$ is also easily inverted on each monomial of the resulting differential polynomial (D on $\widehat{\mathcal{A}}^{[0]}$ just counts the number of variables u^*_*

and ε). The reader can promptly check that we obtain

$$\begin{aligned}
g_{-1} &= u \\
g_0 &= \frac{u^2}{2} + \frac{\varepsilon^2}{24}u_2 \\
g_1 &= \frac{u^3}{6} + \frac{\varepsilon^2}{24}uu_2 + \frac{\varepsilon^4}{1152}u_4 \\
g_2 &= \frac{u^4}{24} + \varepsilon^2\frac{u^2u_2}{48} + \varepsilon^4\left(\frac{7u_2^2}{5760} + \frac{uu_4}{1152}\right) + \varepsilon^6\frac{u_6}{82944}
\end{aligned}$$

The differential polynomials g_i have the property that $\frac{\partial g_i}{\partial u} = g_{i-1}$.

A tau structure is obtained simply by taking $h_i = \frac{\delta \overline{h}_{i+1}}{\delta u}$. Indeed we have $\overline{h}_i = \overline{g}_i$ and tau-symmetry holds. The coordinate u is already a normal coordinate for this tau-structure.

2. Cohomological Field Theories and the double ramification cycle

In this section we introduce the notion of Cohomological Field Theory, a family of cohomology classes on the moduli spaces of stable curves which is compatible with the natural maps and boundary structure [KM94], and the double ramification cycle, another cohomology class representing (a compactification of) the locus of curves whose marked points support a principal divisor. We will assume a certain familiarity with the geometry of the moduli space itself, referring to [Zvo06] for an excellent introductory exposition.

2.1. **Moduli space of stable curves.** Here we just recall the main definitions and fix the notations. Given two integers $g, n \geq 0$ such that $2g-2+n>0$, the moduli space of stable curves will be denoted by $\overline{\mathcal{M}}_{g,n}$. It is a $(3g-3+n)$-dimensional compact complex orbifold (or smooth Deligne-Mumford stack) parametrizing all possible stable Riemann surfaces (stable complex curves) with genus g and n distinct numbered points, up to biholomorphisms preserving nodes and marked points.

On $\overline{\mathcal{M}}_{g,n}$ there is a universal curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, a morphism of orbifolds whose fiber over a point $x \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the curve C_x represented by that point. Each fiber C_x hence has n marked numbered points which, varying $x \in \overline{\mathcal{M}}_{g,n}$, form n sections $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$, $i=1,\ldots,n$.

There are three natural morphisms among different moduli spaces. The forgetful morpshism $\pi : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ forgets the last m marked point on a curve (contracting all components of the curve that might have thus become unstable)

The gluing morphism $\sigma : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ glues two stable curves by identifying the last marked point of the first one with the last marked point of the second one, which become a node.

The loop morphism $\tau : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ identifies the two last marked points on the same stable curve, hence forming a loop and increasing the genus by 1.

On the total space of the universal curve there is a line bundle $\omega_{g,n} \to \overline{\mathcal{C}}_{g,n}$. On the smooth points of the fibers C_x of $\overline{\mathcal{C}}_{g,n}$ it is defined as the relative cotangent (canonical) bundle with respect to the projection $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ and it extends canonically to the singular points to give an actual line bundle on the full $\overline{\mathcal{C}}_{g,n}$.

The tautological bundles $L_i \to \overline{\mathcal{M}}_{g,n}$, $i=1,\ldots,n$ are defined as $L_i = s_i^* \omega_{g,n}$. The fiber of L_i at the point $x \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line at the i-th marked point of the curve C_x represented by x. The first Chern class of L_i will be denoted by $\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

The Hodge bundle $\mathbb{H} \to \overline{\mathcal{M}}_{g,n}$ is the rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $x \in \overline{\mathcal{M}}_{g,n}$ consists of the vector space of abelian differentials on the curve C_x represented by x. Its g Chern classes will be denoted by $\lambda_i = c_i(\mathbb{H}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, $i=1,\ldots,g$, and $\Lambda(s) := \sum_{i=0}^{g} s^i \lambda_i$.

2.2. **Cohomological Field theories.** Cohomological field theories (CohFTs) were introduced by Kontsevich and Manin in [KM94] to axiomatize the properties of Gromov-Witten classes of a given target variety. As it turns out this notion is actually more general, in the sense that not all CohFTs come from Gromov-Witten theory. The main idea is to define a family of cohomology classes on all moduli spaces $\overline{\mathcal{M}}_{g,n}$, for all stable choices of g and n, parametrized by an n-fold tensor product of a vector space, in such a way that they are compatible with the natural maps between moduli spaces we considered above. Let us review their precise definition.

Let $g, n \geq 0$ such that $2g-2+n > 0$. Let V a $\mathbb{C}$-vector space with basis $e_1, \dots, e_N$ and endowed with a symmetric nondegenerate bilinear form η. A cohomological field theory (CohFT) is a system of linear maps $c_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ such that

(i) $c_{g,n}$ is S_n equivariant (with respect to permutations of copies of V in $V^{\otimes n}$ and marked points on the curves),
(ii) $c_{0,3}(e_1 \otimes e_\alpha \otimes e_\beta) = \eta_{\alpha\beta}$,
(iii) $\pi^* c_{g,n}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}) = c_{g,n}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n} \otimes e_1)$ where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$,
(iv) $\sigma^* c_{g_1+g_2,n_1+n_2}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_{n_1+n_2}}) = c_{g_1,n_1+1}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_{n_1}} \otimes e_\mu)\eta^{\mu\nu} c_{g_2,n_2+1}(e_\nu \otimes e_{\alpha_{n_1+1}} \otimes \dots \otimes e_{\alpha_{n_1+n_2}})$ where $\sigma : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$
(v) $\tau^* c_{g+1,n}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}) = c_{g,n+2}(e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n} \otimes e_\mu \otimes e_\nu)\eta^{\mu\nu}$ where $\tau : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$,

The potential of the CohFT is defined as

$$F(t^*_*; \varepsilon) := \sum_{g \geq 0} \varepsilon^{2g} F_g(t^*_*), \quad \text{where}$$

$$F_g(t^*_*) := \sum_{\substack{n \geq 0 \\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\dots,d_n \geq 0} \left\langle \prod_{i=1}^n \tau_{d_i}(e_{\alpha_i}) \right\rangle_g \prod_{i=1}^n t^{\alpha_i}_{d_i},$$

$$\langle \tau_{d_1}(e_{\alpha_1}) \dots \tau_{d_n}(e_{\alpha_n}) \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) \prod_{i=1}^n \psi_i^{d_i}.$$

Some examples of CohFTs are:

- Trivial CohFT: $V = \mathbb{C}$, $\eta = 1$, $c_{g,n} = 1$,
- Hodge CohFT: $V = \mathbb{C}$, $\eta = 1$, $c_{g,n} = \Lambda(s) = \sum_{j=1}^g s^j \lambda_j$
- GW theory of a smooth projective variety X:

  $V = H^*(X, \mathbb{C})$, $\quad \eta =$ Poincaré pairing, $\quad c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) = p_* ev^*(\otimes_{i=1}^n e_{\alpha_i}) q^\beta$

  where $p : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$, $\quad ev : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X^n$ where $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the moduli space of stable maps u from curves C of genus g with n marked points to X of degree $u_*[C] = \beta \in H^2(X, \mathbb{Z})$. The projection p forgets the map u and the evaluation map ev evaluates the map u on the n marked points.
  Notice that, in order to perform the pushforward along p, a notion of Poincaré duality must be used, which involves the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X, \beta)$.
- Witten's r-spin classes:

  $$V = \mathbb{C}^{r-1}, \ r \geq 2, \quad \eta_{\alpha\beta} = \delta_{\alpha+\beta,r}$$

  $c_{g,n}(e_{a_1+1}, \dots, e_{a_n+1}) \in H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ is a class of degree $\frac{(r-2)(g-1)+\sum_{i=1}^n a_i}{r}$ if $a_i \in \{0, \dots, r-2\}$ are such that this degree is a non-negative integer, and vanishes otherwise. The class is constructed in [PV00] (see also [Chi06]) by pushing forward to $\overline{\mathcal{M}}_{g,n}$ Witten's virtual class on the moduli space of curves with r-spin structures. An r-spin structure on a smooth curve $(C, x_1, \ldots, x_n)$ is an r-th root L of the (twisted) canonical bundle $K(\sum a_i x_i)$ of the curve, where $a_i \in \{0, \ldots, r-1\}$. Witten's class is the virtual class of r-spin structures with a holomorphic section (and vanishes when one of the a_i's equals $r-1$), but we will not go into the details of the construction here. This is an example of CohFT that is not a Gromov-Witten theory.

- Fan–Jarvis–Ruan–Witten (FJRW) theory: consider the data of (W, G) where
  - $W = W(y_1, \ldots, y_m)$ is a quasi-homogeneous polynomial with weights $w_1, \ldots, w_m$ and degree d, which has an isolated singularity at the origin,
  - G is a group of diagonal matrices $\gamma = (\gamma_1, \ldots, \gamma_m)$ leaving the polynomial W invariant and containing the diagonal matrix $\mathfrak{j} := (e^{\frac{2\mathrm{i}\pi w_1}{d}}, \ldots, e^{\frac{2\mathrm{i}\pi w_m}{d}})$.

  The vector space V is given by

$$V = \bigoplus_{\gamma \in G} (\mathcal{Q}_{W_\gamma} \otimes d\underline{y}_\gamma)^G,$$

  where W_γ is the γ-invariant part of the polynomial W, $\mathcal{Q}_{W_\gamma}$ is its Jacobian ring, the differential form $d\underline{y}_\gamma$ is $\bigwedge_{y_j \in (\mathbb{C}^m)^\gamma} dy_j$, and the upper-script G stands for the invariant part under the group G. It comes equipped with a bidegree and a pairing, see [CIR14, Equation (4)] or [PV11, Equation (5.12)].
  Roughly, the cohomological field theory [FJR13, FJR07] is constructed using virtual fundamental cycles of certain moduli spaces of stable orbicurves with one orbifold line bundle L_i for each variable y_i, $i = 1, \ldots, m$, such that for each monomial W_j in W, $W_j(L_1, \ldots, L_k) = K(\sum_{i=1}^n x_i)$, where K is the canonical bundle of the curve and $x_1, \ldots, x_n$ are its marked points.

2.3. **Double ramification cycle.** The double ramification cycle (or DR cycle) $\mathrm{DR}_g(a_1, \ldots, a_n)$ is defined as the push-forward to the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ of the virtual fundamental class of the moduli space of rubber stable maps to $\mathbb{P}^1$ relative to 0 and ∞, with ramification profile (orders of poles and zeros) given by $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, where $\sum_{i=1}^n a_i = 0$. Here "rubber" means we consider maps up to the $\mathbb{C}^*$-action in the target $\mathbb{P}^1$ and a positive/negative coefficient a_i indicates a pole/zero at the i-th marked point, while $a_i = 0$ just indicates an internal marked point (that is not a zero or pole).

We view the DR cycle as a cohomology class in $H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ representing some natural compactification of the locus, inside $\mathcal{M}_{g,n}$, formed by complex curves with marked points $x_1, \ldots, x_n$ such that $\sum_{i=1}^n a_i x_i$ is the divisor of the zeros and poles of a meromorphic function.

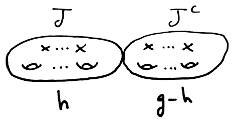
Recently Pixton conjectured an explicit formula for the DR cycle in terms of ψ-classes and boundary strata, which was then proven in [JPPZ16]. The problem of expressing the DR cycle in terms of other tautological classes has been known since around 2000 as Eliashberg's problem, since Yakov Eliashberg posed it as a central question in symplectic field theory, and Pixton's formula provides a surprisingly explicit answer. We will not recall the full formula here, limiting ourselves to recalling instead that the class $DR_g(a_1, \ldots, a_n)$ belongs to $H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, is tautological, and is a (non-homogeneous) polynomial class in the a_i's formed by monomials of even degree and top degree equal to $2g$.

In fact, the restriction of the DR cycle to the moduli space of curves of compact type $\mathcal{M}^{\mathrm{ct}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ (i.e. those stable curves having only separating nodes) is described by the simpler Hain's formula [Hai13]

$$H^{2g}(\mathcal{M}^{\mathrm{ct}}_{g,n}) \ni DR_g(a_1,\dots,a_n)|_{\mathcal{M}^{\mathrm{ct}}_{g,n}} = \frac{1}{g!}\left(-\frac{1}{4}\sum_{J\subset\{1,\dots,n\}}\sum_{h=0}^{g} a_J^2 \delta_h^J\right)^g$$

where

$$a_J := \sum_{j\in J} a_j\ , \qquad \delta_h^J = \left\{ \right.$$



$$\left. \right\}, \qquad \delta_0^{\{i\}} = -\psi_i.$$

From this formula it is apparent that $DR_g(a_1,\dots,a_n)|_{\mathcal{M}^{\mathrm{ct}}_{g,n}}$ is a polynomial class in the a_i's homogeneous of degree $2g$. This formula is useful for instance when computing the intersection in $\overline{\mathcal{M}}_{g,n}$ of $DR_g(a_1,\dots,a_n)$ with the class λ_g, since the latter vanishes outside $\mathcal{M}^{\mathrm{ct}}_{g,n}$ anyway.

3. The Dubrovin-Zhang hierarchy of a cohomological field theory

3.1. **DZ hierarchy.** Dubrovin and Zhang [DZ05], but see also [BPS12a, BPS12b], give a construction of an integrable hierarchy starting from a semisimple cohomological field theory. A CohFT is said to be semisimple when the associative algebra with structure constants $\eta^{\alpha\mu}\frac{\partial F_0}{\partial t^\mu_0 \partial t^\beta_0 \partial t^\gamma_0}\Big|_{t^*_{>0}=0}$ is semisimple generically with respect to the variables t^*_0.

Consider the potential $F(t^*_*;\varepsilon) = \sum_{g\geq 0}\varepsilon^{2g}F_g(t^*_*)$ of the CohFT. Denote $\Omega_{\alpha,p;\beta,q}(t^*_*;\varepsilon) = \frac{\partial^2 F(t^*_*;\varepsilon)}{\partial t^\alpha_p \partial t^\beta_q} = \sum_{g\geq 0}\Omega^{[2g]}_{\alpha,p;\beta,q}(t^*_*)\varepsilon^{2g}$.

The construction starts in genus 0 and we use variables v^*_* for the fomal loops space. Here the hierarchy is given by the following Hamiltonian densities and Poisson structure:

$$\begin{cases} h_{\alpha,p}(v^*) = \Omega^{[0]}_{\alpha,p+1;1,0}(t^*_0 = v^*, 0, 0, \ldots) \\ (K^{\mathrm{DZ}}_v)^{\alpha\beta} = \eta^{\alpha\beta}\partial_x \end{cases}.$$

Commutativity of these Hamiltonians is a simple consequence of the fact that the nodal divisors $D_{(12|34)}$ and $D_{(13|24)}$ are equivalent in $H^*(\overline{\mathcal{M}}_{0,4},\mathbb{Q})$. Also, these Hamiltonian densities are a tau-structure by definition.

Let then $v^\alpha(x,t^*_*)$, $\alpha, 1,\ldots,N$, be the solution to the above integrable hierarchy with initial datum $v^\alpha(x,t^*_*=0) = \delta^\alpha_1 x$. We have, see e.g. [BPS12a],

$$F_g(t^*_0, t^*_1, \ldots) = F_g(P^*_0(v^*_0,\ldots,v^*_{3g-2}),\ldots,P^*_{3g-2}(v^*_0,\ldots,v^*_{3g-2}),0,\ldots)|_{x=0}$$

where P^*_* are in general rational functions, not differential polynomials.

Consider the change of coordinates

$$w^\alpha(v^*_*;\varepsilon) = v^\alpha + \sum_{g\geq 1}\varepsilon^{2g}\frac{\partial^2 F_g(v^*_0,\ldots,v^*_{3g-2})}{\partial t^\alpha_0 \partial x}$$

It is not a Miura transformation, because the P^*_* are not differential polynomials.

The full Dubrovin-Zhang (DZ) hierarchy is just the transformation of the above genus 0 hierarchy with respect to the above non-Miura change of coordinates. In fact, in order to obtain a tau-structure, we want to add a ∂_x-exact term to the Hamiltonians, as prescribed for a normal (albeit non-Miura) transformation:

$$\begin{cases} h^{\mathrm{DZ}}_{\alpha,p}(w^*_*;\varepsilon) := h_{\alpha,p}(w^*_*;\varepsilon) + \sum_{g\geq 1}\varepsilon^{2g}\frac{\partial^2 F_g(w^*_*;\varepsilon)}{\partial t^\alpha_{p+1}\partial x} \\ (K^{\mathrm{DZ}}_w)^{\alpha\beta} = (L^*)^\alpha_\mu \circ (K^{\mathrm{DZ}}_v)^{\mu\nu} \circ L^\beta_\nu \end{cases}$$

where $(L^*)^\alpha_\mu = \sum_{s\geq 0}\frac{\partial w^\alpha}{\partial v^\mu_s}\partial^s_x$, $L^\beta_\nu = \sum_{s\geq 0}(-\partial_x)^s \circ \frac{\partial w^\beta}{\partial v^\nu_s}$.

The DZ hierarchy is an integrable tau-symmetric hierarchy whose tau-function for the solution with initial datum $w^\alpha(x,t^*_*=0;\varepsilon) = \delta^\alpha_1 x$ (called the topological solution) is, *by construction*, the partition function of the CohFT.

The technical hypothesis of semisimplicity of the CohFT is used in the proof that, in spite of the fact that the transformation $v^* \mapsto w^*$ is not Miura, the Hamiltonian densities $h^{\mathrm{DZ}}_{\alpha,p}(w^*_*;\varepsilon)$ and Poisson structure $(K^{\mathrm{DZ}}_w)^{\alpha\beta}$ are still of the correct differential polynomial class.

3.2. **Witten's conjecture and its generalizations.** In [Wit91], Witten conjectured that the partition function of the trivial CohFT is the tau-function of the topological solution to the KdV hierarchy.

Another way to state this, in light of the last section, is that the DZ hierarchy of the trivial CohFT is the KdV hierarchy.

This conjecture was proved by Kontsevich in [Kon92] and, after that, many similar conjectures and results appeared in the literature, consisting in identifying and controlling the DZ hierarchy of a given CohFT. For instance in [FSZ10], Faber-Shadrin-Zvonkine proved that the DZ hierarchy of Witten's r-spin class (for $r \geq 2$ a CohFT that was defined in [PV00]) coincides with the r-KdV Gelfand-Dickey hierarchy, another well known tau symmetric integrable system.

4. Frobenius manifold mirror symmetry

4.1. **Frobenius manifolds.** Frobenius manifolds were introduced by Dubrovin [Dub96] to axiomatize the structure of certain families of $2D$ topological field theories.

A Frobenius manifold is a smooth or analytic (or even formal) N-dimensional manifold M whose tangent spaces T_pM have a structure of Frobenius algebras (a commutative associative algebra with a unit e and a symmetric non-degenerate bilinear form η such that $\eta(u, v \cdot w) = \eta(u \cdot v, w)$, $u, v, w \in T_pM$), smoothly or analytically depending on the point $p \in M$. Moreover we require η to be a flat metric, $\nabla e = 0$ (∇ being the Levi-Civita connection for η) and, if $c(u, v, w) := \eta(u \cdot v, w)$, $\nabla_z c(u, v, w)$ to be symmetric with respect to all 4 vector fields u, v, w, z.

Locally any Frobenius manifold corresponds to a smooth or analytic solution $\mathbb{F}$ (called the Frobenius potential) of the Witten-Dijkgraaf-Verlinde-Verlinde equations

$$\frac{\partial^3 \mathbb{F}}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 \mathbb{F}}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 \mathbb{F}}{\partial t^\gamma \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 \mathbb{F}}{\partial t^\nu \partial t^\alpha \partial t^\delta}$$

$$\frac{\partial^3 \mathbb{F}}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$$

where η is a constant symmetric nondegenerate matrix.

Given $\mathbb{F}(t^*)$, $\eta_{\alpha\beta}$ is the flat metric written in the flat coordinates $t^1, \ldots, t^N$, the structure constants of the Frobenius algebra are $c^\alpha_{\beta\gamma} = \eta^{\alpha\mu} \frac{\partial^3 \mathbb{F}}{\partial t^\mu \partial t^\beta \partial t^\gamma}$ and $e = \frac{\partial}{\partial t^1}$ is the unit.

One often requires the existence of a covariantly linear vector field E, $\nabla(\nabla E) = 0$, called the Euler vector field, whose flow rescales the Frobenius algebra and acts by conformal transformations on the η. This amounts to quasi-homogeneity of the Frobenius potential, $E(\mathbb{F}) = d_F \mathbb{F} +$ (quadratic, linear and constant terms in t^*). In this case we call the Frobenius manifold quasi-homogeneous.

4.2. **Mirror symmetry.** Frobenius manifold have very different origins. Let us enumerate some examples:

- Genus 0 part of cohomological field theories: restricting a CohFT partition function to genus 0 and no psi classes, i.e.
$$\mathbb{F}(t^*) = F_0(t^*_*)|_{\substack{t^*_{>0}=0\\ t^*_0=t^*}},$$
we get a solution to the WDVV equations, which in this case are just a restatement of the fact that the nodal divisors $D_{(12|34)}$ and $D_{(13|24)}$ are equivalent in $H^*(\overline{\mathcal{M}}_{0,4},\mathbb{Q})$.

  This makes the underlying vector space V a Frobenius manifold where η is the (constant) metric of the CohFT, the unit vector field e is the unit of the CohFT, etc.

- Miniversal deformations of simple singularities: a construction due to K. and M. Saito (see for instance Hertling's book [Her02] for a review) endows the base space of a semiuniversal unfolding of an isolated hypersurface singularity with the structure of a Frobenius manifold.

  Without going into details, we can very roughly say that, given an holomorphic hypersurface singularity $\lambda : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and the space M of its semiuniversal deformations λ_t, $t \in M$, we can identify the Jacobian ring $\mathcal{J}(\lambda_t) = \frac{\mathcal{O}_{\mathbb{C}^n,0}}{\langle \partial_{x^1}\lambda_t,\dots,\partial_{x^n}\lambda_t\rangle}$ with th tangent space T_tM, which is hence endowed with the structure of a commutative associative algebra with unit.

  Endowing these tangent spaces with a flat metric is more difficult. Saito proved that special volume forms, called primitive forms $\omega = u(x,t)dx^1 \dots dx^n$, exist such that the residue metric
$$\eta(\partial_{t^\alpha}, \partial_{t^\beta}) = \mathrm{res}_{d\lambda_t=0} \frac{(\partial_{t^\alpha}\lambda_t)(\partial_{t^\beta}\lambda_t)}{(\partial_{x^1}\lambda_t)\dots(\partial_{x^n}\lambda_t)}\omega$$
makes M a Frobenius manifold.

- Landau-Ginzburg models: various generalizations of Saito's construction exist where the germ of the isolated singularity λ is replaced by a more general function on some smooth algebraic variety.

  For instance Dubrovin [Dub96] gave a construction of a Frobenius manifold structure on the Hurwitz space of pairs (C, λ), where C is a smooth algebraic curve of genus g and λ is a meromorphic function with fixed branching profile over ∞. The space of semiuniversal deformations is hence replaced by the space of all possible pairs (C, λ) and the analogue of the theory of primitive forms is developed.

  Similar constructions exist for special classes of more general functions. For instance Sabbah [Sab99] identified a class of functions with isolated singularities on affine manifolds, called tame functions, for which he was able to generalize most of Saito's constructions.

  In this context, all these construction are referred to by the broad name of Landau-Ginzburg models. The function over whose deformation space the Frobenius structure is constructed is called Landau-Ginzburg potential or superpotential.

- Orbit space of Coxeter groups and generalizations: Dubrovin showed in [Dub93] that one can endow the orbit space of a finite Coxeter group, i.e. a finite group of linear transformations of a Euclidean space V generated by reflections, with a structure of Frobenius manifold. This structure is closely related to the one we have seen above for isolated hypersurface singularities (indeed, the orbit space of an irreducible Coxeter group is bi-holomorphically equivalent to the universal unfolding of a simple singularity). This construction was later generalized to extended affine Weyl groups in [DZ98].

In this context we can give the following (admittedly vague) definition of mirror symmetry. Two geometric objects (target manifolds for Gromov-Witten theory, intrinsically defined CohFTs, Landau-Ginzburg models, etc.) are mirror partners when they give rise to the same (isomorphic) Frobenius manifolds.

For instance the genus 0 restriction of Witten's r-spin CohFT, the hypersurface singularity $\lambda(x) = x^r$ and the Coxeter goup A_{r-1} produce, for $r \geq 2$, the same Frobenius manifolds and hence form a mirror symmetric triple [Dub93, Dub96].

The special case $r = 2$ is particularly simple and the reader can check explicitly that the first two Frobenius manifolds coincide. Witten's 2-spin calss is just the trivial CohFT. Its genus 0, the primary partition function is hence $\mathbb{F} = \frac{t^3}{6}$. The simple singularity $\lambda = x^2$ has semiuniversal deformation given by $\lambda_t(x) = x^2 + t$, so the algebra structure is $\partial_t \cdot \partial_t = \partial_t$ and the residue metric (here the primitive form is simply $\omega = 2dx$) is just $\eta(\partial_t, \partial_t) = 1$.

Givental and Teleman have shown that it is possible to reconstruct (up to tensoring with the Hodge CohFT) in an essentially unique way a semisimple cohomological field theory using only information in genus 0 and, in fact, only the corresponding structure of Frobenius manifold. This means that, at least in the semisimple case, we could reformulate the above notion of mirror symmetry, as an equivalence of CohFTs at all genera, with different geometric origins.

Even in the non-semisimple case, when we have two constructions of CohFTs from different geometric objects, we can upgrade the above notion of Frobenius manifold mirror symmetry to higher genus by requiring that the two CohFTs coincide. This is what is conjectured to happen, for instance, for the Fan-Jarvis-Ruan-Witten theory of $(W, \langle \mathfrak{j} \rangle)$ where W is the quintic polynomial $W(y_*) = y_1^5 + \cdots + y_5^5$ and the Gromov-Witten theory of the quintic hypersurface in $\{W(y_*) = 0\} \subset \mathbb{P}^4$. In genus 0 this was proven in [CR10].

5. Integrable systems in Gromov-Witten theory and mirror symmetry

Starting from my thesis I've been interested in Gromov-Witten invariants, often with the aim of better understanding mirror symmetry and its relation with integrable systems. Here we consider the (non-homological) formulation of mirror symmetry as an isomorphism between Frobenius manifolds of different origins: the quantum cohmology of a target variety [KM94], singularity theory of Landau-Ginzburg models [Her02, FJR13], certain algebraic objects like Coxeter or Weyl groups [Dub93, DZ98], the dispersionless limit of an integrable hierarchy [DZ05], etc.

5.1. **Quantum cohomology of orbicurves and their mirror model.** One of my first papers on the subject, [Ros10a], deals with the computation of the quantum cohomology of orbicurves, i.e. closed Riemann surfaces with a finite number of singular points with orbifold structure of order n (stabilizer $\mathbb{Z}/n\mathbb{Z}$). Using techniques I had developed in [Ros08a, Ros08b] for the case of smooth curves (consisting in interpreting the degeneration formulae in relative Gromov-Witten theory in the language of quantum integrable systems arising in Symplectic Field Theory) I was able to classify all orbicurves C whose quantum cohomology has Frobenius potential which is polynomial in the variables t^k, $k = 1, \dots, N-1$ and e^{t^N}, where $\phi_1, \dots, \phi_N$ is a basis for the Chen-Ruan orbifold cohomology $H^*_{\mathrm{CR}}(C, \mathbb{C})$ and $\phi_N \in H^2_{\mathrm{CR}}(C, \mathbb{C})$.

Theorem 5.1. [Ros10a] *The only orbicurves with Frobenius potential polynomial in t^k, $k = 1, \dots, N-1$ and e^{t^N} are those with positive Euler characteristic, i.e. projective lines with at most three orbifold points of the type:*

$$\mathbb{P}^1_{p,q,r}, \qquad \text{with } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

For these orbicurves I was able to compute explicitly the quantum cohomology Frobenius manifold and I found the following example of triple mirror symmetry.

Theorem 5.2. [Ros10a] *The quantum cohomology of projectuve line $\mathbb{P}^1_{p,q,r}$ with three orbifold points of order p, q and r, the Landau-Ginzburg model $\lambda(x, y, z) = -xyz + x^p + y^q + z^r$ and the extended affine Weyl groups of type ADE give rise to the same Frobenius structure for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.*

Notice that, for the case $r = 1$, i.e. the case of "footballs" $\mathbb{P}^1_{p,q}$, this result reduces to mirror symmetry with the Landau-Ginzburg model of Laurent polynomials $\lambda(x) = x^p + t^1 x^{p-1} + \ldots + t^{p+q} x^{-q}$ studied in [MT08]. This theorem proves in particular a conjecture of Takahashi [Tak10], who predicted the Landau-Ginzburg model (although I was unaware of this at the time) from homological mirror symmetry considerations.

More recently, Milanov, Shen and Tseng [MST14] where able to identify the Dubrovin-Zhang hierarchy associated to the Gromov-Witten theory of orbicurves $\mathbb{P}^1_{p,q,r}$ with positive Euler characteristic, as the Kac-Wakimoto hierarchies of type ADE (or ADE-Toda hierarchies).

5.2. **Quantum cohomology of local $\mathbb{P}^1$ and its mirror model.** A similar result was obtained more recently in a collaboration with A. Brini and G. Carlet. We identified a mirror partner of the local projective line (i.e. the Gromov-Witten target space modeling the local geometry of a neighborhood of a rational curve in a Calabi-Yau threefold). This target space is actually non-compact and can be described as the rank two bundle $O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1$. Gromov-Witten theory is only defined in its equivariant version using localization with respect to an appropriate (anti-diagonal) $\mathbb{C}^*$ action on the fibres of this bundle (in particular degree 0 curves form non-compact moduli spaces, and the $\mathbb{C}^*$ action is used to localize the computation to compact fixed loci).

Moreover we found that the corresponding Dubrovin-Zhang integrable hierarchy at genus 0 is the dispersionless Ablowitz-Ladik hierarchy.

Theorem 5.3 ([BCR12])**.** *The equivariant quantum cohomology of $O_{\mathbb{P}^1}(-1)\oplus O_{\mathbb{P}^1}(-1)$ and the Landau-Ginzburg model $\lambda(z) = e^v z \frac{z-e^q}{z-e^{-q}}$, $v, q \in \mathbb{C}$ give rise to the same Frobenius manifold. The corresponding dispersionless Dubrovin-Zhang hierarchy is the dispersionless Ablowitz-Ladik hierarchy.*

5.3. **Quantum cohomology of local footballs and their mirror model.** In [BCRR14] we recently managed to understand how to generalize the above result to local projective lines with two orbifold points (toric local $\mathbb{P}^1$-orbifolds). In doing this we discovered a new family of integrable hierarchies parametrized by two positive integer parameters. They all consist in very peculiar reductions of the famous 2-dimensional Toda hierarchy.

2Toda is an integrable system with two space and one time variables (a $2+1$ system), hence it lives on a bigger phase space than the ones we have considered so far. We found in particular that its Poisson structure degenerates on a two non-negative integer parameters family of Poisson submanifolds, giving rise to completely kinematic reductions (which means that this reductions do not depend on the form of the 2Toda Hamiltonians, but just on the form of its Poisson structure). We called these rational reductions, or $RR2T_{a,b}$ because of the special form of their Lax operator.

From the point of view of integrable systems, this family subsumes various other hierarchies of different origins, namely the extended bigraded Toda hierarchy (involved in the Gromov-Witten theory of $\mathbb{P}^1$-orbifolds), the Ablowitz-Ladik hierarchy (GW theory of local $\mathbb{P}^1$), and the q-deformed Gelfand-Dickey hierarchies.

From the point of view of Gromov-Witten theory and mirror symmetry we related the local footballs with a generalization of Dubrovin's Frobenius structure on Hurwitz spaces (called double Hurwitz spaces, as they involve branched covers of $\mathbb{P}^1$ with fixed ramification profile over two points). In particular we have the following result.

Theorem 5.4 ([BCRR14])**.** *The equivariant quantum cohomology of the local footballs $O_{\mathbb{P}^1_{a,b}}(-1/a) \oplus O_{\mathbb{P}^1_{a,b}}(-1/b)$ for $a, b \geq 1$ and the double Hurwitz spaces of meromorphic functions $\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ with fixed ramification profile $(a, 1, \ldots, 1)$ over 0 and $(b, 1, \ldots, 1)$ over ∞ carry isomorphic Frobenius structures. The corresponding integrable hierarchy is the dispersionless limit of the rational reductions $RR2T_{a,b}$.*

This suggests the following Witten-type conjecture.

Conjecture 5.5 ([BCRR14])**.** *The all genus Gromov-Witten potential of the local football $O_{\mathbb{P}^1_{a,b}}(-1/a) \oplus O_{\mathbb{P}^1_{a,b}}(-1/b)$, $a, b \geq 1$ is the topological tau-function of the rational reduction $RR2T_{a,b}$*

In [BCRR14] we successfully tested this conjecture in genus 1 for any $a, b \geq 1$.

6. Double ramification hierarchies

In this section I review one of my main lines of research, a joint project with A. Bruryak and, more recently, with J. Guéré and B. Dubrovin. It deals with a novel construction that associates an integrable, tau-symmetric hierarchy to a cohomological field theory (this time without the semisemplicity assumption which is needed for the Dubrovin-Zhang hierarchy) and is inspired by Eliashberg, Givental and Hofer's Symplectic Field Theory [EGH00].

Since the construction makes explicit use of the intersection theory of the double ramification cycle, we call this hierarchy the double ramification (DR) hierarchy. In its classical version it was introduced by A. Buryak in [Bur15], who also explicitly computed the first two examples (the classical DR hierarchies of the trivial and Hodge CohFTs, corresponding to the KdV and Intermediate Long Wave hierarchies), thereby showing the interest and power of this technique.

Its properties, quantization and relation with the DZ hiearchy were studied and clarified in the series of joint papers [BR16a, BR16b, BDGR16a, BDGR16b], partly guided by our previous investigations of the classical and quantum integrable systems arising in SFT [FR11, Ros15].

The DR hierarchy has many interesting properties and even advantages over the more classical Dubrovin-Zhang hierarchy, including a much more direct access to the explicit form of the Hamiltonians and Poisson structure, a natural and completely general technique to quantize the integrable systems thus produced, recursion relations for the Hamiltonians that are reminiscent of genus 0 TRRs in Gromov-Witten theory but work at all genera. When Dubrovin proposed to me to work on a thesis on integrable systems arising in SFT, back in 2004, he said he believed that was the actual correct approach to integrable hierarchies from moduli spaces of curves. I believe that prediction has found complete confirmation in the power of the DR hierarchy project.

Finally, one of the main parts of this project is the proof of the conjecture (originally proposed in a weaker form by A. Buryak) that the DZ and DR hierarchies for a semisimple CohFT are in fact equivalent under a normal Miura transformation that we completely identified in [BDGR16a]. While the general proof of such conjecture is the object of an ongoing work, we managed to show its validity in a number of examples and classes of interesting special cases. Our present approach to the general statement reduces it to proving a fnite number of relations in the tautological ring of each $\overline{\mathcal{M}}_{g,n}$.

6.1. **The main idea.**

Symplectic Field Theory [EGH00] is a large project attempting to provide a unified view on established pseudoholomorphic curve theories in symplectic topology like symplectic Floer homology, contact homology and Gromov-Witten theory, leading to numerous new applications, including a construction of quantum integrable systems from the geometry of the moduli spaces of pseudoholomorphic curves in symplectic cobordisms between contact manifolds.

In a sense, the double ramification hierarchy arises from completely analogous constructions in the complex algebraic setting and with the axiomatized language of cohomological field theories replacing curve counting in target varieties. In this sense the double ramification hierarchy is a quantum integrable system, even if A.

Buryak introduced first its classical version in [Bur15].

Given a cohomological field theory $c_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$, at the heart of the construction for the classical hierarchy lie its intersection numbers with the DR cycle, the powers of one psi-class and the top Hodge class λ_g:

$$P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n} = \int_{DR_g(-\sum a_i, a_1,\ldots,a_n)} \lambda_g \psi_1^d c_{g,n+1}\left(e_\alpha \otimes \otimes_{i=1}^n e_{\alpha_i}\right).$$

This is all the geometric content used in the definition of the DR hierarchy.

These intersection numbers are collected into generating functions $g_{\alpha,d}$ depending on the indices $\alpha = 1,\ldots,N$ and $d \geq 0$ which have the form of differential polynomials (see next section). The differential polynomials $g_{\alpha,d}$ directly play the role of Hamiltonian densities for a classical integrable system. The Poisson structure, on the other hand, and contrary to what happens for the DZ hierarchy, does not depend on the cohomological field theory and is always in Getzler's standard form.

Notice that, because of the presence of the class λ_g, Hain's formula is sufficient to compute the above intersection numbers. This advantage if often exploited in explicit computations.

6.2. **DR hierarchy Hamiltonians.** Because of the polynomiality properties of the DR cycle, $P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}$ is a homogeneous polynomial in $a_1,\ldots,a_n$ of degree $2g$. So, if we write it as such,

$$P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n} = \sum_{\sum b_i = 2g} \widetilde{P}^{g;b_1,\ldots,b_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}\, a_1^{b_1}\ldots a_n^{b_n},$$

we can give the following definition:

$$\begin{aligned} g_{\alpha,d} &:= \sum_{\substack{g\geq 0, n\geq 0\\ 2g-1+n>0}} \frac{(-\varepsilon^2)^g}{n!} \sum_{a_1,\ldots,a_n\in\mathbb{Z}} P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}\, p^{\alpha_1}_{a_1}\ldots p^{\alpha_n}_{a_n}\, e^{ix\sum a_i} \\ &= \sum_{\substack{g\geq 0, n\geq 0\\ 2g-1+n>0}} \frac{(-\varepsilon^2)^g}{n!} \sum_{\sum b_i = 2g} \widetilde{P}^{g;b_1,\ldots,b_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}\, u^{\alpha_1}_{b_1}\ldots u^{\alpha_n}_{b_n} \end{aligned}$$

and hence we have two expressions for the DR Hamiltonian densities, in variables p^*_* and u^*_* respectively. The second line, in particular, is clearly a differential polynomial in $\widehat{\mathcal{A}}^{[0]}$.

Commutativity $\{\overline{g}_{\alpha,p}, \overline{g}_{\beta,q}\} = 0$ with respect to the standard Hamiltonian operator $(K^{\mathrm{DR}})^{\mu\nu} = \eta^{\mu\nu}\partial_x$ (we omit the subscript K in $\{\cdot,\cdot\}_K$ when K is in Getzler's standard form), was proved in [Bur15]. Let's give an idea of the proof.

In genus 0, where the DR cycle is equal to 1, this equation is basically equivalent to the equivalence of boundary divisors $D_{(12|34)}$ and $D_{(13|24)}$ in $H^*(\overline{\mathcal{M}}_{0,4}, \mathbb{Q})$. The genus 0 argument can be ported to higher genus by working with images of the curves of the DR cycle with respect to the meromorphic function (or more precisely rubber map to $\mathbb{P}^1$) that is defined on them. This is a general fact: we often find that genus 0 properties of the DZ hierarchy have all genera analogues on the DR hierarchy side.

Making this argument precise, one gets to prove the following equation for products of double ramification cycles. For a subset $I = \{i_1, i_2, \ldots\}, i_1 < i_2 < \ldots$, of the set $\{1,\dots,n\}$ let $A_I := (a_{i_1}, a_{i_2}, \dots)$. Suppose the set $\{1,2,\dots,n\}$ is divided into two disjoint subsets, $I \sqcup J = \{1,2,\dots,n\}$, in such a way that $\sum_{i\in I} a_i > 0$. Let us denote by $DR_{g_1}(0_{x_1}, A_I, -k_1, \dots, -k_p) \boxtimes DR_{g_2}(0_{x_2}, A_J, k_1, \dots, k_p)$ the cycle in $\overline{\mathcal{M}}_{g_1+g_2+p-1,n+2}$ obtained by gluing the two double ramification cycles at the marked points labeled by the positive integers $k_1,\dots,k_p$. Here 0_x indicates a a coefficient 0 at the marked point x. Then

$$\sum \frac{\prod_{i=1}^p k_i}{p!} DR_{g_1}(0_{x_1}, A_I, -k_1, \dots, -k_p) \boxtimes DR_{g_2}(0_{x_2}, A_J, k_1, \dots, k_p) \tag{6.1}$$

$$-\sum \frac{\prod_{i=1}^p k_i}{p!} DR_{g_1}(0_{x_2}, A_I, -k_1, \dots, -k_p) \boxtimes DR_{g_2}(0_{x_1}, A_J, k_1, \dots, k_p) = 0. \tag{6.2}$$

The sum is over I, J, $p > 0$ $k_1 > 0, \dots, k_p > 0$, $g_1 \geq 0$, $g_2 \geq 0$.

If we intersect this relation with the class λ_g (which kills the terms with $p > 1$) and with the ψ-classes and CohFT, and form the corresponding generating function, we obtain precisely

$$\sum_{k>0} \left(k\eta^{\mu\nu} \frac{\partial \overline{g}_{\alpha,p}}{\partial p^\mu_k} \frac{\partial \overline{g}_{\beta,q}}{\partial p^\nu_{-k}} - k\eta^{\mu\nu} \frac{\partial \overline{g}_{\beta,q}}{\partial p^\mu_k} \frac{\partial \overline{g}_{\alpha,p}}{\partial p^\nu_{-k}} \right) = \{\overline{g}_{\alpha,p}, \overline{g}_{\beta,q}\} = 0.$$

In [Bur15] Buryak computed the first two examples of DR hierarchies. For the trivial CohFT he found the KdV hierarchy, the same result as for the DZ hierarchy. For the Hodge CohFT he found the Intermediate Long Wave hierarchy (ILW). When comparing this second case with the DZ hierarchy he realized that, once more, the integrable systems were the same, but this time he had to perform a Miura transformation to match them. This motivated him to propose the following conjecture.

Conjecture 6.1 (Weak DR/DZ equivalence [Bur15])**.** *Given a semisimple CohFT, the associated DZ and DR hierarchy coincide up to a Miura transformation.*

6.3. **Recursion relations.** In [BR16a], using results about the intersection of a ψ-class with the DR cycle from [BSSZ15] by analogy with my previous paper with O. Fabert [FR11], we found the following recursion equations among the DR Hamiltonian densities.

Theorem 6.2 ([BR16b])**.** *For all $\alpha = 1,\dots,N$ and $p = -1,0,1,\dots$, let $g_{\alpha,-1} = \eta_{\alpha\mu}u^\mu$. We have*

$$\partial_x (D-1) g_{\alpha,p+1} = \left\{ g_{\alpha,p}, \overline{g}_{1,1} \right\}, \tag{6.3}$$

$$\partial_x \frac{\partial g_{\alpha,p+1}}{\partial u^\beta} = \left\{ g_{\alpha,p}, \overline{g}_{\beta,0} \right\}, \tag{6.4}$$

where $D := \varepsilon \frac{\partial}{\partial \varepsilon} + \sum_{s\geq 0} u^\alpha_s \frac{\partial}{\partial u^\alpha_s}$.

Equation (6.3) is especially striking. First of all it provides and effective procedure to reconstruct the full hierarchy starting from the knowledge of $\overline{g}_{1,1}$ only. Secondly, from the point of view of integrable systems, such recursion was not known. Even in the simplest examples it does not coincide with any previously known reconstruction techniques for the symmetries of an integrable hierarchy (it is in fact this recursion that we presented in section 1.5 for the KdV equation) . At the same time, its universal form (its form is rigid, independent of the CohFT or the integrable hierarchy) suggests that it should be regarded as some sort of intrinsic feature of at least a class of integrable systems (see section 6.7).

6.4. **Tau-structure and the strong DR/DZ equivalence.** In [BDGR16a] we provide the DR hierarchy with a tau-structure and study its topological tau-function.

Theorem 6.3. *The DR hierarchy is tau-symmetric. A tau-structure is given by* $h_{\alpha,p} = \frac{\delta \overline{g}_{\alpha,p+1}}{\delta u^1}$.

Consider the normal coordinates $\widetilde{u}^\alpha = \eta^{\mu\nu} h_{\mu,-1}$. Let us write the tau-function associated to the topological solution (with initial datum $\widetilde{u}^\alpha(x,0;\varepsilon) = x\delta^\alpha_1$) as

$$F^{\mathrm{DR}}(t^*_*;\varepsilon) = \sum_{g\geq 0} \varepsilon^{2g} F^{\mathrm{DR}}_g(t^*_*), \quad \text{where}$$

$$F^{\mathrm{DR}}_g(t^*_*) = \sum_{\substack{n\geq 0\\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\dots,d_n\geq 0} \left\langle \prod_{i=1}^n \tau_{d_i}(e_{\alpha_i}) \right\rangle^{\mathrm{DR}}_g \prod_{i=1}^n t^{\alpha_i}_{d_i}.$$

Notice that this *DR partition function* has only an indirect geometric meaning. Contrary to the correlators of the topological tau-function of the DZ hierarchy (which coincide with the correlators of the CohFT), the correlators $\langle \prod_{i=1}^n \tau_{d_i}(e_{\alpha_i})\rangle^{\mathrm{DR}}_g$ are not a priori defined as intersection numbers in $H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$, but only as the coefficients of the series F^{DR}. We can a posteriori try to study their geometric meaning, and, as a consequence of certain properties of the DR cycle, we find the following surprising selection rule.

Proposition 6.4 ([BDGR16a])**.** $\langle \tau_{d_1}(e_{\alpha_1})\dots\tau_{d_m}(e_{\alpha_m})\rangle^{\mathrm{DR}}_g = 0$ *when* $\sum_{i=1}^m d_i > 3g-3+m$ *or* $\sum_{i=1}^m d_i \leq 2g-2$.

In light of the conjectured equivalence with the DZ hierarchy, the first selection rule looks like the corresponding vanishing property $\langle \tau_{d_1}(e_{\alpha_1})\dots\tau_{d_m}(e_{\alpha_m})\rangle_g = 0$ when $\sum_{i=1}^m d_i > 3g-3+m$, which just means that we cannot integrate too many ψ-classes without surpassing the dimension of the moduli space (for short, we say that correlators cannot be "too big"). But the second selection rule actually says that the DR correlators cannot be too small either! This rule one has no analogue in the DZ case and, as it turns out, provides the key to a much deeper understanding of the DR/DZ equivalence.

The situation is that we are trying to compare two integrable tau-symmetric hierarchies by a Miura transformation that is supposed to modify the tau-function by killing all "small correlators" (which are present on the DZ side and absent in the DR side). A natural candidate would then be a normal Miura transformation (since they preserve tau-symmetry) generated by a differential polynomial $\mathcal{F}(w^*_*;\varepsilon) \in \widehat{\mathcal{A}}^{[-2]}$,

$$\widetilde{u}^\alpha = w^\alpha + \eta^{\alpha\mu}\partial_x\{\mathcal{F}, \overline{h}^{\mathrm{DZ}}_{\mu,0}\}_{\mathrm{DZ}}$$

and we know that such transformations modify the tau-function by

$$\widetilde{F}(t^*_*;\varepsilon) = F(t^*_*;\varepsilon) + \mathcal{F}(w^*_*(x,t^*_*;\varepsilon);\varepsilon)|_{x=0}$$

Can we find $\mathcal{F}(w^*_*;\varepsilon)$ so that $\widetilde{F}(t^*_*;\varepsilon)$ satisfies the selection rule (i.e. has no small correlators)? As it turns out, yes, and this selects a unique normal Miura trasnformation!

Theorem 6.5 ([BDGR16a])**.** $\exists!\ \mathcal{F}(w^*_*;\varepsilon) \in \widehat{\mathcal{A}}^{[-2]}$ *such that* $F^{\mathrm{red}} := F + \mathcal{F}(w^*_*;\varepsilon)|_{x=0}$ *satisfies the above selction rules.*

This makes Buryak's conjecture much more precise.

Conjecture 6.6 (Strong DR/DZ equivalence, [BDGR16a])**.** *For any semisimple CohFT, the DR and DZ hierarchies coincide up to the normal Miura transformation generated by the unique $\mathcal{F}(w^*_*;\varepsilon)$ found in Theorem 6.5. Even in the non-semisimple case, we can state this conjecture as $F^{\mathrm{red}} = F^{\mathrm{DR}}$.*

When proven true, the conjecture would clearly state that, although equivalent as integrable systems to the DZ hiearchy, the DR hiearchy contains strictly less information than the DZ hiearchy. Indeed, starting from the DZ hierarchy it is possible to construct the normal Miura transformation mapping to the DR hiearchy, while the DR hierarchy does not contain this extra information. This is perhaps not surprising given at least the presence of the class λ_g in the DR hiearchy intersection numbers.

From the point of view of integrable systems however, this is of great interest. The fact the DR hierarchy is some sort of standard form of the DZ hiearchy allows to study these systems ignoring complications that might just come from the system of coordinates in which they are described. The presence of powerful recursion relations for the Hamiltonians, for instance, seems to rely precisely on this special standard form.

Finally we remark that the extra information that is killed by the above normal Miura transformation, might be (maybe in part) recovered once we consider the quantum DR hiearchy (which replaces λ_g in the construction by the full Hodge class $\Lambda(s)$), see below.

6.5. **The proof of the strong DR/DZ conjecture.** In [BDGR16a] we prove the strong DR/DZ equivalence conjecture for a number of CohFTs.

Theorem 6.7 ([BDGR16a])**.** *The strong DR/DZ equivalence conjecture holds in the following cases:*

- *the trivial CohFT,*
- *the full Hodge class,*
- *Witten's* 3-*,* 4- *and* 5-*spin classes*
- *the GW theory of* $\mathbb{P}^1$*,*
- *up to genus* 5 *for any rank* 1 *CohFT,*
- *up to genus* 1 *for any semisimple CohFT.*

We also have an unpublished proof for the Gromov-Witten theory of any smooth variety of positive dimension with non-positive first Chern class and for the D_4 Fan-Jarvis-Ruan-Witten class [FJR13, FFJMR16] which will appear in an upcoming publication with my student C. Du Crest de Villeneueve.

However, in all these cases, the proof is either by direct computation or by some ad hoc technique. A large and quite technical part of our project deals with proving Conjecture 6.6 on completely general grounds.

The strategy of the proof for the general case which we are pursuing, in [BDGR16b] and our next paper in progress, is to give explicit geometric formulas for the correlators appearing in both F^{DZ} and F^{red} in terms of sums over certain decorated trees corresponding to cycles in the $\overline{\mathcal{M}}_{g,n}$. This reduces the strong DR/DZ conjecture to a family of relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$. In particular we managed to further reduce this family to a finite number (equal to the number of partitions of

$2g$) of relations in each $\overline{\mathcal{M}}_{g,n}$.

6.6. **Quantization.** As we already remarked, the idea for the DR hierarchy came from Symplectic Field Theory where quantum integrable systems arise naturally. Let us see how this happens in the language we used in this document, of cohomological field theories in the complex algebraic category. The intersection numbers to be considered look perhaps more natural,

$$P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}(s) = \int_{DR_g(-\sum a_i,a_1,\ldots,a_n)} \Lambda(s)\,\psi_1^d c_{g,n+1}\left(e_\alpha \otimes \otimes_{i=1}^n e_{\alpha_i}\right).$$

Indeed the product $\Lambda(s)c_{g,n+1}\left(e_\alpha \otimes \otimes_{i=1}^n e_{\alpha_i}\right)$ is itself a CohFT (and every CohFT can be written this way), so we are simply intersecting a CohFT, the ψ-classes and the DR cycle.

$P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}(s)$ is a non-homogeneous polynomial in $a_1,\ldots,a_n$ of top degree $2g$, so

$$P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}(s) = \sum_{\sum b_i=2k\le 2g} \widetilde{P}^{g;b_1,\ldots,b_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}(s)\ a_1^{b_1}\ldots a_n^{b_n}$$

and we define

$$\begin{aligned} G_{\alpha,d} &:= \sum_{\substack{g\ge 0,n\ge 0\\ 2g-1+n>0}} \frac{(i\hbar)^g}{n!} \sum_{a_1,\ldots,a_n\in\mathbb{Z}} P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}\left(\frac{-\varepsilon^2}{i\hbar}\right) p^{\alpha_1}_{a_1}\ldots p^{\alpha_n}_{a_n}\ e^{ix\sum a_i} \\ &= \sum_{\substack{g\ge 0,n\ge 0\\ 2g-1+n>0}} \frac{(i\hbar)^g}{n!} \sum_{\sum b_i\le 2g} \widetilde{P}^{g;b_1,\ldots,b_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}\left(\frac{-\varepsilon^2}{i\hbar}\right) u^{\alpha_1}_{b_1}\ldots u^{\alpha_n}_{b_n} \end{aligned}$$

Notice how $(i\hbar)$ has replaced $(-\varepsilon^2)$ as the genus parameter and, at the same time, we have given the Hodge class parameter s the value $\left(\frac{-\varepsilon^2}{i\hbar}\right)$, so that these two choices compensate in the limit $\hbar = 0$ to give back the classical Hamiltonian densities $g_{\alpha,p}$.

What about commutativity of these new Hamiltonians? We can again use equation (6.1), but, because the top Hodge class λ_g has now been replaced by the full Hodge class $\Lambda(s)$, all values of $p > 0$ will contribute to the sum. This translates into the following equation:

$$[\overline{G}_{\alpha,p}, \overline{G}_{\beta,q}] = 0$$

where $[f,g] := f\star g - g\star f$ with $f\star g = f\left(e^{\sum_{k>0} i\hbar k\eta^{\alpha\beta}\overleftarrow{\frac{\partial}{\partial p^\alpha_k}}\overrightarrow{\frac{\partial}{\partial p^\beta_{-k}}}}\right)g$. The exponential here comes precisely from the fact that double ramification cycles are now glued along any number of marked points, not just one, as it was the case for the classical DR hierarchy.

So we have commutativity in a new sense, namely with respect to the non-commutative product $\star$. This is a star product deforming Getzler's standard Poisson structure in the deformation quantization sense, so what we obtain is a genuine quantization of the classical DR hierarchy. In [BR16b] we actually develop a precise extension of the formal loop space with this quantum algebra structure (including a surprising formula to express the $\star$-product in terms of u^*_* variables) that, we believe, has independent value.

From a mathematical physics viewpoint this is an entirely new and surprisingly universal quantization technique for integrable field theories. We have completely explicit formulas for the quantum versions of KdV, Toda, ILW, Gelfand-Dickey and other integrable hiearchies, that, to our knowledge, were either unknown or known in a much more indirect way.

This explicit description also rests on the anlogue of Theorem 6.2 which, again, allows to reconstruct the full quantum hierarchy from the Hamiltonian $\overline{G}_{1,1}$ alone.

Theorem 6.8 ([BR16b])**.** *For all* $\alpha = 1, \dots, N$ *and* $p = -1, 0, 1, \dots,$ *let* $G_{\alpha,-1} = \eta_{\alpha\mu} u^\mu$*. We have*

$$\partial_x (D-1) G_{\alpha,p+1} = \frac{1}{\hbar} \left[G_{\alpha,p}, \overline{G}_{1,1} \right], \tag{6.5}$$

$$\partial_x \frac{\partial G_{\alpha,p+1}}{\partial u^\beta} = \frac{1}{\hbar} \left[G_{\alpha,p}, \overline{G}_{\beta,0} \right], \tag{6.6}$$

where $D := \varepsilon \frac{\partial}{\partial \varepsilon} + 2\hbar \frac{\partial}{\partial \hbar} + \sum_{s \geq 0} u^\alpha_s \frac{\partial}{\partial u^\alpha_s}$.

Finally, in [BDGR16b], we define and study the quantum analogue of the notion of tau-structure and tau-functions and prove that the quantum DR hierarchy satisfies tau-symmetry. This allows to define a quantum deformation of the DR potential that clearly contains more geometric information on the associated CohFT and needs to be investigated further. For instance, can one recover, from this information, the full DZ hierarchy and, eventually, the tautological part of the CohFT?

6.7. **Integrable systems of DR type.** The recursion equation (6.5) or its classical version (6.3) are really surprising from the point of view of integrable systems. No expert we talked to was able to recognize them as something previously known.

Moreover we realized that one could interpret such equation as constraints for the generating Hamiltonian $\overline{G}_{1,1}$ itself, just by imposing that, at each step of the recursion, we still obtain a commuting quantity. This technique proved fruitful to reproduce, for instance, the full DR hierarchy starting from genus 0 in the case of polynomial Frobenius manifolds (i.e. those genus 0 CohFT associated with Coxeter groups as in [Dub93]). In doing these computational experiments we realized that the recursions (6.5), (6.3) were of independent value in the theory of integrable systems.

For simplicity, let us state our result in the classical situation.

Theorem 6.9. *Assume that a local functional* $\overline{g}_{1,1} \in \widehat{\Lambda}^{[0]}$ *(not necessarily of geometric origin) is such that the recursion* (6.3) *produces, at each step, Hamiltonians that still commute with* $\overline{g}_{1,1}$ *(so that the recursion can go on indefinitely). Assume moreover that* $\frac{\delta \overline{g}_{1,1}}{\delta u^1} = \frac{1}{2} \eta_{\mu\nu} u^\mu u^\nu + \partial_x^2 r$*, where* $r \in \widehat{\mathcal{A}}^{[-2]}$.

Then we have

(i) $g_{1,0} = \frac{1}{2} \eta_{\mu\nu} u^\mu u^\nu + \partial_x^2 (D-1)^{-1} r,$

(ii) $\{\overline{g}_{\alpha,p}, \overline{g}_{\beta,q}\} = 0, \qquad \alpha, \beta = 1, \dots, N, \quad p, q \geq -1,$

(iii) $\{g_{\alpha,p}, \overline{g}_{\beta,0}\} = \partial_x \frac{\partial g_{\alpha,p+1}}{\partial u^\beta}, \qquad \beta = 1, \dots, N, \quad p \geq -1,$

(iv) $\frac{\partial g_{\alpha,p}}{\partial u^1} = g_{\alpha,p-1}, \qquad \alpha = 1, \dots, N, \quad p \geq -1,$

hence in particular we get an integrable tau-symmetric hierarchy.

This suggests that it is interesting to consider integrable systems originating from local functionals satisfying the hypothesis of the above theorem. We call them integrable systems of DR type.

Since the hypothesis above can be easily checked, we were able to give a low genus classification of rank 1 integrable systems of DR type. It turns out that they seem to correspond precisely to rank 1 cohomological field theory. Tests in rank 2 show the emergence of integrable systems of more general origin. However this was expected from geometry too. Indeed the construction of the classical DR hierarchy also works for partial CohFTs, i.e. CohFTs that do not satisfy the loop gluing axiom. It would appear from low genus computations that classical integrable systems of DR type are classified by partial CohFTs but only those coming from actual CohFTs possess a quantization.

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P. Rossi:
IMB, UMR5584 CNRS, Université de Bourgogne Franche-Comté, F-21000 Dijon, France
E-mail address: paolo.rossi@u-bourgogne.fr