Double ramification hierarchies and their applications

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Integrable systems

Evolutionary PDEs :

$$u^\alpha = u^\alpha(x,t), \qquad x \in S^1, \quad \alpha = 1,\dots,N$$

$$\frac{\partial u^\alpha}{\partial t} = F(u, u_x, u_{xx}, \dots)$$

a vector field on the "space" of $u^\alpha(x)$

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Poisson structure :

$$\{\overline{f},\overline{g}\}_K = \sum_{s\geq 0} \frac{\partial f}{\partial u_s^\mu} \partial_x^s \left(K^{\mu\nu} \frac{\delta \overline{g}}{\delta u^\nu} \right)$$

$$K^{\mu\nu} = \sum_{j\geq 1} K_j^{\mu\nu} \partial_x^j, \qquad K_j^{\mu\nu} \in \mathcal{A}^{[-j+1]}$$

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Coordinate (Miura) tranformations :

$$\widetilde{u}^\alpha = \widetilde{u}^\alpha(u^*_*, \varepsilon) \in \mathcal{A}^{[0]}$$

$$\det\left(\frac{\partial \widetilde{u}^*|_{\varepsilon=0}}{\partial u^*}\right) \neq 0$$

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Action on Poisson bracket :

$$K^{\alpha\beta}_{\widetilde{u}} = (L^*)^\alpha_\mu \circ K^{\mu\nu}_u \circ L^\beta_\nu$$

$$(L^*)^\alpha_\mu = \sum_{s\geq 0} \frac{\partial \widetilde{u}^\alpha}{\partial u^\mu_s}\partial_x^s, \qquad L^\beta_\nu = \sum_{s\geq 0}(-\partial_x)^s \circ \frac{\partial \widetilde{u}^\beta}{\partial u^\nu_s}$$

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Theorem (Getzler '02)

There exist a Miura transformation bringing any Poisson bracket to the standard form

$$K^{\mu\nu} = \eta^{\mu\nu}\partial_x, \qquad \eta^{\mu\nu} \text{ constant, symmetric and nondegenerate}$$

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Hamiltonian system : $$\frac{\partial u^\alpha}{\partial t} = \{u^\alpha, \overline{h}\}_K, \qquad h(u^*_*;\varepsilon) \in \Lambda^{[0]}$$

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Example (KdV) :

$$N = 1, \qquad K = \partial_x$$

$$\overline{h}_{\mathrm{KdV}} = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx} \right) dx$$

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$$\overline{h}_0 = \int \left(\frac{u^2}{2} \right) dx$$

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx} \right) dx$$

$$\overline{h}_2 = \int \left(\frac{u^4}{24} + \frac{\varepsilon^2}{24} \left(u u_x^2 + u^2 u_x x \right) + \frac{\varepsilon^4}{480} u u_{xxxx} \right) dx$$

$$\dots$$

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Tau functions : Let $\partial_x \Omega_{\alpha,p;\beta,q}(u^*_*;\varepsilon) := \{h_{\alpha,p-1}, \overline{h}_{\beta,q}\}_K$.

Then for any solution $u^*(x,t,\varepsilon)$ there exists $F(t^*_*,\varepsilon)$

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Normal coordinates : $\widetilde{u}^\alpha = \eta^{\alpha\mu} h_{\mu,-1}(u^*_*;\varepsilon)$

$K^{\alpha\beta}_{\widetilde{u}} = \eta^{\alpha\beta}\partial_x + O(\varepsilon), \qquad \eta$ const. sym. nondeg.

Integrable systems

Normal Miura transf. : (a Miura that preserves the tau structure)

Let u^α already be normal coordinates

and let $\mathcal{F}(u^*_*;\varepsilon) \in \mathcal{A}^{[-2]}$:

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Effect on tau functions : $\widetilde{F}(t^*_*;\varepsilon) = F(t^*_*;\varepsilon) + \mathcal{F}(u^*_*(x,t^*_*;\varepsilon);\varepsilon)|_{x=0}$

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$$[f,\overline{g}] = \sum_{\substack{n\geq 1\\ r_1,\ldots,r_n\geq 0\\ s_1,\ldots,s_n\geq 0}} \frac{(-i)^{n-1}\hbar^n}{n!} \frac{\partial^n f}{\partial u_{s_1}^{\alpha_1}\ldots\partial u_{s_n}^{\alpha_n}} (-1)^{\sum_{k=1}^n r_k} \left(\prod_{k=1}^n \eta^{\alpha_k\beta_k}\right) \times$$

$$\times \sum_{j=1}^{2n-1+\sum(s_k+r_k)} (-1)^{\frac{2n-1+\sum(s_k+r_k)-j}{2}} C_j^{s_1+r_1+1,\ldots,s_n+r_n+1} \partial_x^j \frac{\partial^n g}{\partial u_{r_1}^{\beta_1}\ldots\partial u_{r_n}^{\beta_n}}.$$

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$$\mathrm{Li}_{-d_1}(z)\dots\mathrm{Li}_{-d_k}(z) = C_j^{d_1,\dots,d_k}\mathrm{Li}_{-j}(z), \qquad \mathrm{Li}_{-d}(z) := \sum_{k\geq 0} k^d z^k.$$

Moduli space of curves

$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{c} \text{closed stable Riemann surfaces of genus } g, \\ \text{with } n \text{ distinct marked points } x_1, \ldots, x_n \end{array} \right\} \Big/ \sim$$

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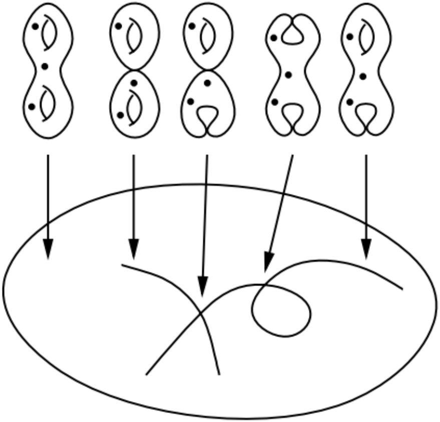
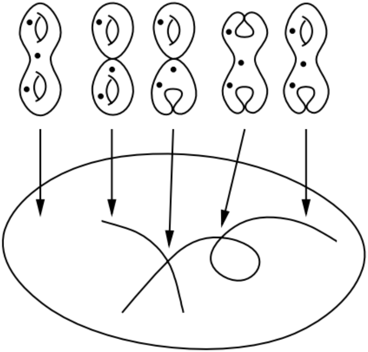
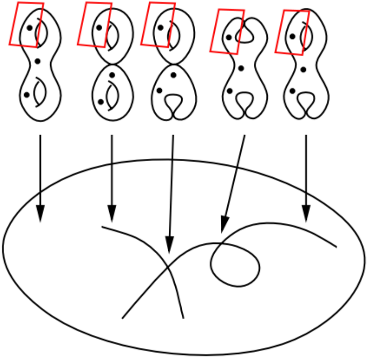


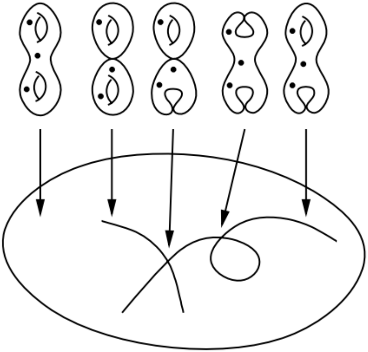
$\overline{\mathcal{M}}_{g,n}$ is compact of dimension $3g - 3 + n$ [Deligne,Mumford, '69]





Tautological bundles :

$$L_i \to \overline{\mathcal{M}}_{g,n}, \quad i = 1, \ldots, n$$



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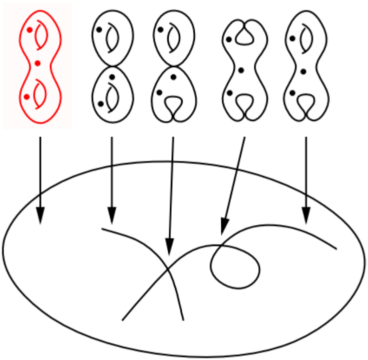
$\psi_i = c_1(L_i) \ \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

$\langle \phi_1, \ldots, \phi_g \rangle$



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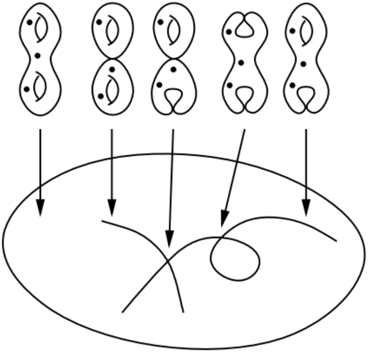

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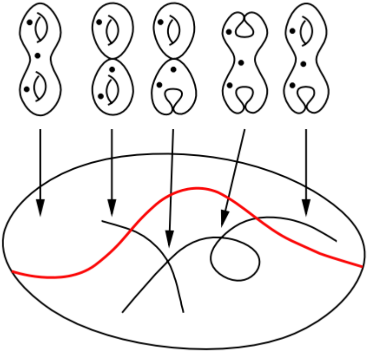

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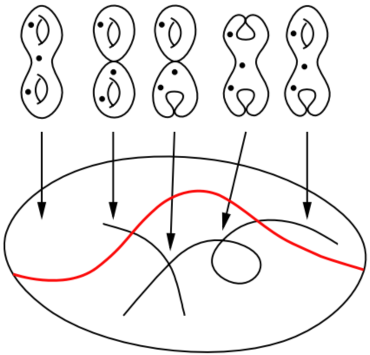
Double ramification cycle :

$DR_g(a_1, \dots, a_n) = \{(C; x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n} \mid \exists f, (f) = \sum a_i x_i\} \subset \overline{\mathcal{M}}_{g,n}$

$\dim DR_g(a_1, \dots, a_n) = 2g - 3 + n, \qquad a_i \in \mathbb{Z}$



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$$DR_g(a_1, \ldots, a_n) = PD\left(p_* \left[\overline{\mathcal{M}}_{g,(a_1,\ldots,a_n)}(\mathbb{P}^1; 0, \infty)\right]^{\mathrm{vir}}\right) \in H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

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Hain's formula :

$$\overline{\mathcal{M}}_{g,n} \supset \mathcal{M}^{ct}_{g,n} = \{\mathcal{C} \in \overline{\mathcal{M}}_{g,n} | \text{ all nodes of } C \text{ are separating}\}$$

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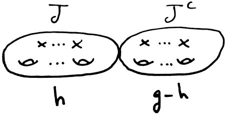
$$H^{2g}(\mathcal{M}^{ct}_{g,n}) \ni DR_g(a_1, \ldots, a_n)|_{\mathcal{M}^{ct}_{g,n}} = \frac{1}{g!}\left(-\frac{1}{4}\sum_{J\subset\{1,\ldots,n\}}\sum_{h=0}^{g} a_J^2 \delta_h^J\right)^g$$

$$a_J := \sum_{j\in J} a_j \ , \qquad \delta_h^J = \left\{ \text{[diagram]} \right\}, \qquad \delta_0^{\{i\}} = -\psi_i$$



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Pixton's formula :

$$P_g^r(a_1,\dots,a_n) := \sum_{\substack{\Gamma \text{ stable graph} \\ w \text{ weighting mod } r}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{h^1(\Gamma)} \xi_{\Gamma *} \left[\prod_{i=1}^{n} e^{a_i^2 \psi_i} \prod_{e=(h,h')\in E(\Gamma)} \frac{1-e^{-w(h)w(h')(\psi_h+\psi_{h'})}}{\psi_h+\psi_{h'}} \right]$$

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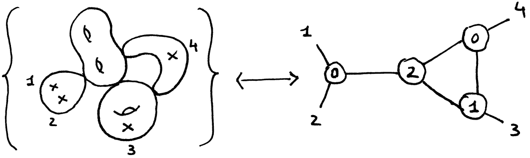
Stable graphs : $\overline{\mathcal{M}}_\Gamma =$  $= \Gamma$

Double ramification cycle

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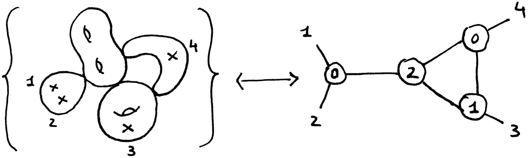
Stable graphs : $\overline{\mathcal{M}}_\Gamma =$ 

$= \Gamma$

Weightings mod r : $w : H(\Gamma) \to \{0,\dots,r-1\}$, such that

$$w(h_i) = a_i \bmod r, \qquad w(h) + w(h') = 0 \bmod r$$

$$\sum_{h \text{ at fixed vertex}} w(h) = 0 \bmod r$$

Double ramification cycle

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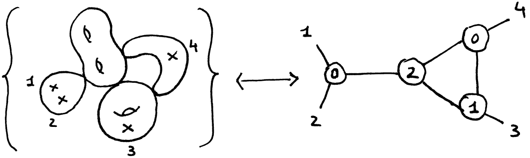
Stable graphs : $\quad \overline{\mathcal{M}}_\Gamma = \Big\{$  $= \Gamma$

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Potential :

$$F(t^*_*; \varepsilon) := \sum_{g \geq 0} \varepsilon^{2g} F_g(t^*_*), \quad \text{where}$$

$$F_g(t^*_*) := \sum_{\substack{n \geq 0 \\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\dots,d_n \geq 0} \left\langle \prod_{i=1}^n \tau_{d_i}(e_{\alpha_i}) \right\rangle_g \prod_{i=1}^n t^{\alpha_i}_{d_i},$$

$$\langle \tau_{d_1}(e_{\alpha_1}) \dots \tau_{d_n}(e_{\alpha_n}) \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) \prod_{i=1}^n \psi_i^{d_i}.$$

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- Witten's r-spin classes : $V=\mathbb{C}^{r-1}$, $\eta^{\alpha\beta}=\delta_{\alpha+\beta,r}$

$$c_{g,n}^{r-\mathrm{spin}}(e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_n})=(-1)^d r^{1-g}p_*c_{g,n}^{\mathrm{vir}}(\alpha_1,\ldots,\alpha_n)$$

where $c_{g,n}^{\mathrm{vir}}(\alpha_1,\ldots,\alpha_n)\in H^*(\mathcal{S}_{g,n}^r(\alpha_1,\ldots,\alpha_n))$

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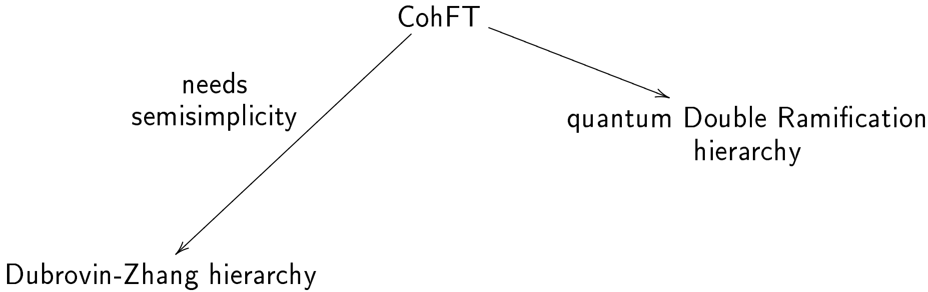
Integrable hierarchies from CohFTs



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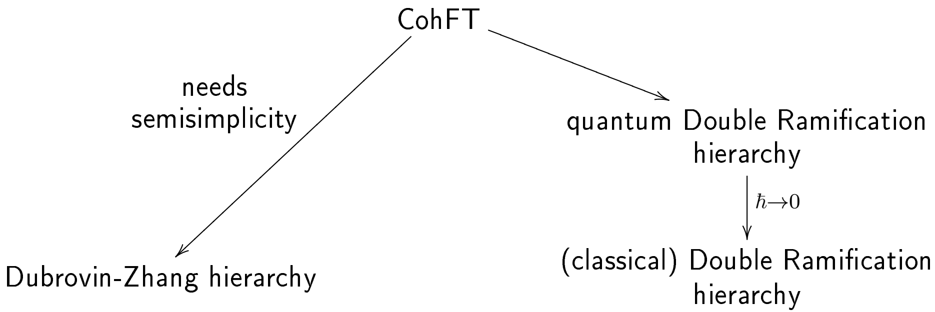

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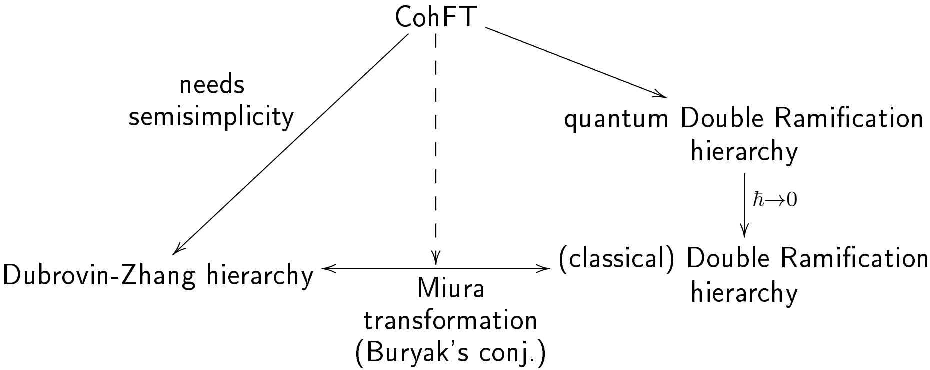

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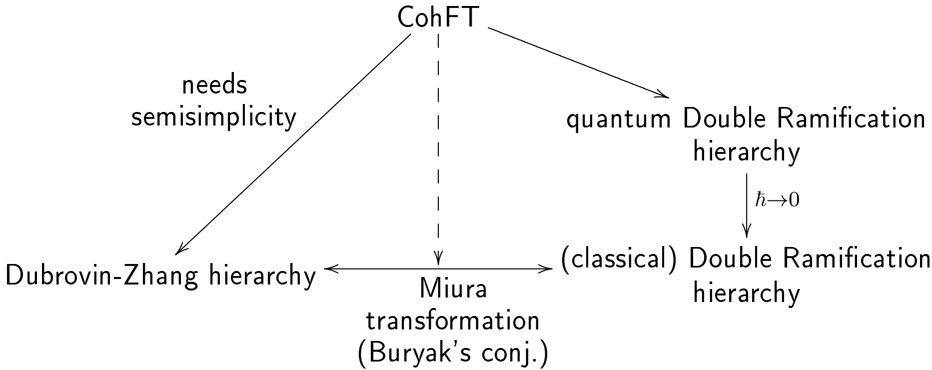

DZ hierarchy : the central object is $F(t^*_*;\varepsilon)$, which is interpreted as the (log of the) tau function of the topological ($u^\alpha(x,0;\varepsilon) = x\delta^\alpha_1$) solution to the system of PDEs. Hamiltonians and Poisson structure can be reconstructed from it.

Dubrovin-Zhang hierarchy of a semisimple CohFT

$$F(t_0^*,t_1^*,\ldots;\varepsilon)=\sum_{g\geq 0}F_g(t_0^*,t_1^*,\ldots)\varepsilon^{2g},\qquad \Omega_{\alpha,p;\beta,q}(t_0^*,t_1^*,\ldots;\varepsilon)=\frac{\partial^2 F(t_0^*,t_1^*,\ldots;\varepsilon)}{\partial t_p^\alpha \partial t_q^\beta}$$

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Genus 0 : $\begin{cases} \overline{h}^{[0]}_{\alpha,p}(v^*) = \Omega^{[0]}_{\alpha,p+1;1,0}(t^*_0 = v^*, 0, 0, \dots) \\ (K^{\mathrm{DZ}}_v)^{\alpha\beta} = \eta^{\alpha\beta}\partial_x \end{cases}$

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Proposition (Eguchi-Getzler-Xiong, DZ, Buryak-Posthuma-Shadrin)

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$$w^\alpha(v_*^*; \varepsilon) = v^\alpha + \sum_{g \geq 1} \varepsilon^{2g} \frac{\partial^2 F_g(v_0^*, \ldots, v_{3g-2}^*)}{\partial t_0^\alpha \partial x}$$ not a Miura transf. !

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$h^{\mathrm{DZ}}_{\alpha,p}(w_*^*;\varepsilon):=h^{[0]}_{\alpha,p}(w_*^*;\varepsilon)+\sum_{g\geq 1}\varepsilon^{2g}\frac{\partial^2 F_g(w_*^*;\varepsilon)}{\partial t_{p+1}^\alpha \partial x}$

$F(t_*^*;\varepsilon)$ tau function for $w^\alpha(x,0;\varepsilon)=x\delta_1^\alpha$

Double ramification hierarchy

$$P^{g;a_1,\dots,a_n}_{\alpha,d;\alpha_1,\dots,\alpha_n} = \int_{DR_g(-\sum a_i,a_1,\dots,a_n)} \lambda_g \psi_1^d c_{g,n+1}\left(e_\alpha \otimes \otimes_{i=1}^n e_{\alpha_i}\right)$$

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$$g_{\alpha,d} := \sum_{\substack{g\geq 0,n\geq 0\\ 2g-1+n>0}} \frac{(-\varepsilon^2)^g}{n!} \sum_{a_1,\ldots,a_n\in\mathbb{Z}} P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}\ p^{\alpha_1}_{a_1}\ldots p^{\alpha_n}_{a_n}\ e^{ix\sum a_i}$$

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where $u^\alpha_b = \partial_x^b\left(\sum_{a\in\mathbb{Z}} p^\alpha_a e^{ibx}\right)$

Quantum double ramification hierarchy

Diff. polynomials : $\mathcal{A} := \mathbb{C}[[u_0^*]][u_*^*][[\varepsilon]], \quad \deg \varepsilon = -1, \quad \deg u_k^* = k$

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$$P^{g;a_1,\ldots,a_n}_{\alpha,d;\alpha_1,\ldots,\alpha_n}(s) = \int_{DR_g(-\sum a_i,a_1,\ldots,a_n)} \Lambda(s)\,\psi_1^d c_{g,n+1}\left(e_\alpha \otimes \otimes_{i=1}^n e_{\alpha_i}\right)$$

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$$G_{\alpha,d} := \sum_{\substack{g\geq 0, n\geq 0\\ 2g-1+n>0}} \frac{(i\hbar)^g}{n!} \sum_{a_1,\dots,a_n\in\mathbb{Z}} P^{g;a_1,\dots,a_n}_{\alpha,d;\alpha_1,\dots,\alpha_n}\left(\frac{-\varepsilon^2}{i\hbar}\right) p^{\alpha_1}_{a_1}\dots p^{\alpha_n}_{a_n}\; e^{ix\sum a_i}$$

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where $u^\alpha = \sum_{a\in\mathbb{Z}} p^\alpha_a e^{ibx}$

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Classical double ramification hierarchy

Historically this came first, invented in [Buryak '14].

Classical limit : $\frac{1}{\hbar}[F, \overline{G}]\big|_{\hbar=0} = \{\cdot,\cdot\}$ Poisson bracket in std form

$$G_{\alpha,p}(u^*_*;\varepsilon,\hbar)|_{\hbar=0} =: g_{\alpha,p}(u^*_*;\varepsilon) \in \mathcal{A}^{[0]}$$

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Conjecture (DR/DZ equivalence, Buryak '14)

There exists a Miura transformation $w^\alpha \mapsto u^\alpha$ such that

$$\overline{h}^{\mathrm{DZ}}_{\alpha,p}(w^*_*;\varepsilon) \mapsto \overline{g}_{\alpha,p}(u^*_*;\varepsilon) \qquad \{\cdot,\cdot\}^{\mathrm{DZ}} \mapsto \{\cdot,\cdot\}$$

Double ramification tau structure

DR tau structure : $h_{\alpha,p} := \frac{\delta \overline{g}_{\alpha,p}}{\delta u^1} \implies \begin{cases} \overline{h}_{\alpha,p} = \overline{g}_{\alpha,p} \\ \{h_{\alpha,p-1}, \overline{h}_{\beta,q}\} = \{h_{\beta,q-1}, \overline{h}_{\alpha,p}\} \end{cases}$

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DR tau function : tau function associated to $\widetilde{u}^\alpha(x,0;\varepsilon) = x\delta^\alpha_1$:

$$F^{\mathrm{DR}}(t^*_*;\varepsilon) =: \sum_{g\geq 0} \varepsilon^{2g} F^{\mathrm{DR}}_g(t^*_*), \quad \text{where}$$

$$F^{\mathrm{DR}}_g(t^*_*) =: \sum_{\substack{n\geq 0\\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\ldots,d_n\geq 0} \left\langle \prod_{i=1}^n \tau_{d_i}(e_{\alpha_i}) \right\rangle^{\mathrm{DR}}_g \prod_{i=1}^n t^{\alpha_i}_{d_i}.$$

Strong DR/DZ equivalence

Proposition (Buryak-Dubrovin-Guéré-R. '16)

$$\langle \tau_{d_1}(e_{\alpha_1}) \dots \tau_{d_m}(e_{\alpha_m}) \rangle_g^{\mathrm{DR}} = 0$$

when $\sum_{i=1}^m d_i > 3g-3+m$ *or* $\sum_{i=1}^m d_i \le 2g-2.$

Remark : the first selection rule holds for $F(t^*_*;\varepsilon)$ too, but not the second !

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Idea : We know that normal Miura transformations of the form

$$\widetilde{u}^\alpha = w^\alpha + \eta^{\alpha\mu} \partial_x \{\mathcal{F}, \overline{h}^{\mathrm{DZ}}_{\mu,0}\}^{\mathrm{DZ}}$$

generated by $\mathcal{F}(w^*_*, \varepsilon) \in \mathcal{A}^{[-2]}$, changes tau functions by

$$\widetilde{F}(t^*_*; \varepsilon) = F(t^*_*; \varepsilon) + \mathcal{F}(w^*_*(x, t^*_*; \varepsilon); \varepsilon)|_{x=0}$$

Can we find $\mathcal{F}(w^*_*; \varepsilon)$ so that $\widetilde{F}(t^*_*; \varepsilon)$ satisfies the selection rule ?

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Theorem (BDGR '16)

$\exists! \ \mathcal{F}(w^*_*; \varepsilon) \in \mathcal{A}^{[-2]}$ such that $F^{\mathrm{red}} := F + \mathcal{F}(w^*_*; \varepsilon)|_{x=0}$ satisfies the above selction rules.

Conjecture (Strong DR/DZ conjecture)

The DR and DZ hierarchies are equivalent via the normal Miura transformation generated by the unique $\mathcal{F}(w^*_*; \varepsilon)$ found above ($\iff F^{\mathrm{red}} = F^{\mathrm{DR}}$).

Strong DR/DZ equivalence

Theorem (B '14, BR '14, BG'15, BDGR '16)

The strong DR/DZ equivalence conjecture holds for the trivial CohFT, the full Hodge class, Witten's 3-, 4- and 5-spin classes and the GW theory of $\mathbb{P}^1$.

Remark : for degree reasons, if the conjecture is true for the ADE Fan-Jarvis-Ruan-Witten CohFTs, the normal Miura transformation is trivial.

Quantization of important integrable hierarchies

- KdV :

$$\overline{G}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx} - \frac{i\hbar}{24} u \right) dx$$

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Generating series for $\varepsilon = 0$: $\sum_{d \geq -1} y^d \, G_d|_{\varepsilon=0} = \frac{1}{y^2 S(\sqrt{i}\lambda y)} e^{yS\left(\frac{\lambda}{\sqrt{i}} y \partial_x\right) u} - y^{-2}$

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- Toda :

$$\overline{G}_{1,1} = \int \left(\frac{(u^1)^2 u^\omega}{2} + \sum_{g \geq 1} \varepsilon^{2g} \frac{B_{2g}}{(2g)!} u^1 u^1_{2g} + q \left(\frac{e^{\frac{\varepsilon \partial_x}{2}} + e^{-\frac{\varepsilon \partial_x}{2}}}{2} u^\omega - 2 \right) e^{S(\varepsilon \partial_x) u^\omega} \right.$$

$$\left. + q u^\omega - \frac{i\hbar}{12} u^1 + i\hbar \sum_{g \geq 1} \varepsilon^{2g-2} \frac{B_{2g}}{(2g)!} u^\omega_{2g} u^1 \right) dx.$$

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- Invariant part of a CohFT wrt a finite symmetry group Γ :

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- Even part of GW theory :

Quintic threefold :

$$\bar{g}_{1,1} = \int \left(\frac{1}{2}(u^1)^2 u^4 + \frac{5}{6}(u^2)^3 + u^1 u^2 u^3 + \sum_{d=1}^{\infty} c_d (du^2 - 2)\ q^d e^{du^2} + \frac{25\varepsilon^2}{3}(u^1_x)^2 \right) dx$$

DRH for smooth varieties with $c_1(X) \leq 0$

Proposition

Let X be a smooth variety with $\dim X > 0$ and non-positive first Chern class. Let $1, \theta_1, \dots, \theta_M$ be a homogeneous basis for $H^{\text{even}}(X, \mathbb{Q})$ (hence with $\deg \theta_i \geq 2$). Then

$$g_{\alpha,p} = g^{[0]}_{\alpha,p} + \delta_{\alpha,1} \frac{\varepsilon^2}{24} \frac{\chi(X)}{p!} (u^1)^p \; u^1_{xx},$$

where $\chi(X)$ is the Euler characteristic of X and u^1 is the variable associated with the class 1.

Examples : all Calabi-Yau manifolds, surfaces of general type, Enriques surfaces, degree D hypersurfaces in $\mathbb{CP}^N$ with $D > N > 1$, etc.

DRH for smooth varieties with $c_1(X) \leq 0$

Examples :

- $c_1(X) < 3 - \dim X \implies \overline{g}_{1,1} = \int \left(\int_X \frac{\theta^3}{3!} - \varepsilon^2 \frac{\chi(X)}{24} (u_x^1)^2 \right) dx$, e.g. $K3$ surfaces ($c_1(X) = 0,\ \chi(X) = 24,\ \dim H^{\mathrm{ev}}$ = 24) :

$$\overline{g}_{1,1} = \int \left(\int_X \frac{\theta^3}{3!} - \varepsilon^2 (u_x^1)^2 \right) dx$$

- smooth hypersurfaces of degree D in $\mathbb{P}^N$ with $c_1(X) = (N+1-D)H \leq 0$, e.g quintic ($c_1(X) = 0,\ \chi(X) = -200,$ $\dim H^{\mathrm{ev}}$ = 4).
- Enriques surface $X = \frac{K3}{\sigma}$ ($2c_1(X) = 0,\ \chi(X) = 12,\ \dim H^{\mathrm{ev}}$ = 12) :

$$\overline{g}_{1,1} = \int \left(\int_X \frac{\theta^3}{3!} - \varepsilon^2 \frac{(u_x^1)^2}{2} \right) dx$$

- Enriques CY3 $X = \frac{K3 \times E}{(\sigma, -1)}$ ($c_1(X) = 0,\ \chi(X) = 0,\ \dim H^{\mathrm{ev}}$ = 24) :

$$\overline{g}_{1,1} = \overline{g}_{1,1}^{[0]}$$

Classification of hierarchies of DR type

The recursion relation

$$\begin{cases} G_{\alpha,d+1} = (D-1)^{-1}\partial_x^{-1}[G_{\alpha,d}, \overline{G}_{1,1}] \\ G_{\alpha,-1} = \eta_{\alpha\mu}u^\mu \end{cases}$$

(maybe together with string equation $\frac{\partial G_{\alpha,d+1}}{\partial u^1} = G_{\alpha,d}$ or the more general $\frac{\partial G_{\alpha,d+1}}{\partial u^\beta} = \partial_x^{-1}[G_{\alpha,d}, \overline{G}_{\beta,0}]$) can be used to look for all $\overline{G}_{1,1}$ that produce commuting hierarchies.

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<u>Fact</u> :
In $\dim V = 1$ at low genus ($g \leq 7$) we find a family of $\overline{G}_{1,1}$ with one new parameter at each genus (even at $\hbar = 0$). These parameters correspond to the most general CohFT (a product of any number of Hodge classes $\Lambda(s) = \sum s^i\lambda_i$).

<u>Fact</u> : In $\dim V = 2$ and $\hbar = 0$ we start finding partial CohFTs.

Computing Gromov-Witten invariants by degeneration

The degeneration formula for (relative) GW invariants can be used together with the DR hierarchy to compute GW invariants. In the simple case of $\mathbb{P}^1 \times \mathbb{P}^1$ it looks something like :

$$e^{F_{\mathbb{P}^1\times\mathbb{P}^1}(t^{(1,1)},t^{(\omega,1)},s_d^{(1,\omega)},s_d^{(\omega,\omega)};\varepsilon,\hbar)} = \Bigg[e^{F_{(\mathbb{P}^1\times\mathbb{P}^1,D_\infty)}\left(\frac{\partial}{\partial p^1_{>0}},\frac{\partial}{\partial p^\omega_{>0}};\varepsilon,\hbar\right)} e^{\sum s_d^\alpha \overline{G}_{\alpha,d}\left(p^1_{\geq 0},p^\omega_{\geq 0},\frac{\partial}{\partial p^1_{>0}},\frac{\partial}{\partial p^\omega_{>0}};\varepsilon,\hbar\right)} e^{F_{(\mathbb{P}^1\times\mathbb{P}^1,D_0)}(p^1_{>0},p^\omega_{>0};\varepsilon,\hbar)} \Bigg]_{p^*_*=0}$$

where $\alpha = (1,\omega),(\omega,\omega)$ and $p_0^1 = t^{(1,1)}$, $p_0^\omega = t^{(\omega,1)}$.

Changing the caps on the left/right with more general objects (for instance blowups) one gets other surfaces (see [Cooper-Pandharipande, Block-Goettsche]).

But we can compute any descandants (recursion) and insert λ-classes !

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